LEVEL II

NUSC Technical Report 6015

(12)

Singular Integrals and Their Evaluation

Phillip B. Abraham
Special Projects Department

22 January 1979

NUSC
NAVAL UNDERWATER SYSTEMS CENTER
Newport, Rhode Island • New London, Connecticut

PREFACE

This work was conducted under Project No. A-126-11, "Interarray Processing," Principal Investigator, J. B. Hall (Code 3211), Program Element 63504 SO222-AS, Naval Sea Systems Command, Program Manager, R. T. Cockerell (Code SEA-06H2).

The present work has benefited from several discussions between the author and Dr. A. H. Nuttall (Code 313), to whom the author is pleased to express his sincere thanks. Dr. Nuttall also served as the technical reviewer for this report.

Reviewed and Approved: 22 January 1979

R. W. Hasse
Head: Special Projects Department

The author of this report is located at the
New London Laboratory, Naval Underwater Systems Center,
New London, Connecticut 06320.

(14) NUSC-

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
TR-6Ø15

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER
(9) Technical Rept.

4. TITLE (and Subtitle)
(6) SINGULAR INTEGRALS AND THEIR EVALUATION

5. TYPE OF REPORT & PERIOD COVERED

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
(10) Phillip B. Abraham

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Naval Underwater Systems Center
New London Laboratory
New London, CT 20360

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
A1261 (16)
63504 SØ222-AS

11. CONTROLLING OFFICE NAME AND ADDRESS
Naval Sea Systems Command
(SEA-06H2-4)
Washington, DC 20362

12. REPORT DATE
(11) 22 Jan[redacted]979

13. NUMBER OF PAGES
60

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
(12) 63p.

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(17) SØ222AS

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this work, integrals with simple or multiple poles which are located on the path of integration are systematically treated. It is shown that with each such integral an infinite set of values can be associated. Since the Principal Value, introduced by Cauchy, plays a key role in the subject, a detailed discussion of this concept is included, and a variety of methods for its explicit evaluation are presented. Various Plemelj formulas are also derived. An important feature, that may well be novel in this context, is

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20. (Cont'd)

a δ-function interpretation of the associated integrands, and its subsequent implications. Examples, related to cases frequently met in applications, are treated in detail in several appendices.

TABLE OF CONTENTS

I. INTRODUCTION

Various physical and engineering problems require the evaluation of integrals, over the real line, of complex functions that have singularities, such as poles and branch points, along the path of integration. These integrals, which appear frequently when solving differential or integral equations by integral transform methods, ordinarily arise when the inverse transforms need to be evaluated.

As is well-known, such integrals may have no meaning at all in the sense of Riemann's theory of integration. The frustrated user of the integral transform method may at this point become discouraged, if not despairing, and forsake this approach for more conventional, though less direct, methods.

This state of affairs is common to other areas of human activity and may well stem from the implicit faith one has in the power of a particular discipline to produce automatically the correct results when applied to the problems of a different discipline. Mathematics in itself, when legitimately applied, will yield not only the correct result, but also the entire class of all possible results to the problem under consideration. This clearly implies that we cannot live by mathematics alone. Our salvation lies then in choosing that solution, from the set of all mathematically possible ones, which satisfies the basic principles of the discipline which gave rise to the original problem. To illustrate, it is well-known that waves cannot propagate inside a quiescent medium prior to the activation of a source of disturbances. Nevertheless, one of the mathematical solutions to the wave equation appropriate to this problem is a wave propagating before the disturbances have been initiated. This wave, known as the advanced solution, is usually discarded, since it does not conform with our perception of physical causality, i.e., that effect should not precede cause. Examples abound in the physical and engineering literature where certain possible solutions are discarded because these violate basic requirements of the various theories to which the problems belong.

In the present report, the author proposes to list the possible mathematical interpretations of the integrals mentioned above, restricted at present only to poles along the path of integration, none of which occur at an endpoint of the interval of integration.

We start this report in Section II with a discussion of integrals having simple poles at a finite distance from the origin, the interval of integration being the whole real axis. We shall call these full-range integrals. In three subsections we proceed to give three interpretations, that increase in generality, as follows: The Principal Value of Cauchy, the Plemelj formulas obtained by path deformation, and the δ-function interpretation which includes the first two as special cases. It is of interest to note that the third interpretation has not been sufficiently emphasized in the literature, although it is based on well-known results of generalized function theory (the solution of equations with vanishing coefficients, or the zero divisor problem).

In Section III we consider full-range integrals possessing multiple poles on the real axis. The results of Section II are extended, with the appropriate interpretation, also to this case. The need for a more general definition of the Principal Value is shown by exhibiting the failure of the conventional one to exist for a simple example.

Similar results are presented in Sections IV and V for half-range integrals and finite intervals. In the treatment are included both simple and multiple poles. In these sections several methods are proposed for the explicit evaluation of the Principal Value, while these and other methods are described in Appendix I for the full range integrals. Some of these methods may be novel. The fact of the matter is that the literature on the subject is rather silent about effecting the evaluation of the Principal Value by methods which avoid the ε-limit process. As can be seen from the text, the Principal Value, when it exists, is made the cornerstone of all the possible interpretations of a singular integral and thus it is essential to have an explicit expression for it, whether in closed form or as an ordinary, convergent integral.

In Appendix II the theory is applied to specific examples for full-range integrals with simple poles. We mention in particular the results obtained when there is an infinite number of poles. The method used is of interest in itself, being essentially an extension, to certain classes of functions, of the partial fraction decomposition formulas for polynomials (Ref. 10). Since infinite sums occur in this case, questions of convergence are raised. It is not surprising, perhaps, that these can be bypassed by an appeal to the theory of generalized functions. This theory also provides the interpretation when the full-range integrals do not converge, even with the poles removed. Thus, although the Principal Value does not exist in the conventional sense, it does so as a generalized function. This aspect of the Principal Value does not seem to have been highlighted in previous works.

In Appendix III one example of a full-range integral with a multiple pole is presented. Appendices IV and V treat examples for half-range and finite intervals, respectively, with simple poles. Examples for multiple poles have not been given for these cases, but their treatment presents no particular difficulty.

In the concluding remarks, a short outline of the main theme of the work is given. Several related, important topics, not investigated here, which require a similar systematic treatment, are mentioned.

II. INTEGRALS WITH SIMPLE POLES

In this section we shall consider integrals of the type

$$I(\xi,\lambda) = \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\,dx \tag{1}$$

In eq. (1), $f(x,\lambda)$ is a complex function of the real variables x and λ. The latter can be viewed as a varying parameter, independent of x. In addition, ξ is real and $f(x,\lambda)$ is defined for $x=\xi$, with $f(\xi,\lambda) = 0$. Hence the integrand in $I(\xi,\lambda)$ has a real pole of first order at $x=\xi$.

$I(\xi,\lambda)$, as it stands, represents three limit processes, each of which may yield a divergent result. From Analysis we know that eq. (1) is a shorthand notation for

$$I(\xi,\lambda) = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b \frac{f(x,\lambda)}{x-\xi}\,dx \tag{2}$$

where $a \to -\infty$ and $b \to +\infty$ <u>independently</u>. The integral over the finite interval a,b represents the third limit process (one of summation in this case) alluded to above. Let us denote this integral by I_a^b. Due to the presence of the pole at $x=\xi$, this integral does not exist in the conventional sense.

In the following we introduce three interpretations, of increasing generality, which allow us to associate specific values to singular integrals of the type under consideration.

a. <u>Principal Value Interpretation</u>

Such integrals were considered, among others, by Cauchy, who was the first to propose one interpretation, to which he gave the name "Valeur Principale", or Principal Value (PV), now commonly denoted by the letter P. The definition is

$$\text{Principal Value of } I_a^b \equiv P\int_a^b \frac{f(x,\lambda)}{x-\xi}\,dx \equiv$$

$$= \lim_{\substack{\varepsilon\to 0 \\ \varepsilon>0}} \left\{ \int_a^{\xi-\varepsilon} \frac{f(x,\lambda)}{x-\xi}\,dx + \int_{\xi+\varepsilon}^b \frac{f(x,\lambda)}{x-\xi}\,dx \right\} \tag{3}$$

with $a < \xi < b$.

The conditions for the existence of the PV are:

i) $f(x,\lambda)$ is integrable in the interval a,b
ii) as $\varepsilon\to 0$ through positive values, the limit of the expression inside the curly brackets exists, even though the limits of the individual terms may fail to do so.

The PV integral of eq. (3) has been investigated by Hilbert, who looked upon it as the integral transform, with kernel $(x-\xi)^{-1}$, of $f(x,\lambda)$. Hilbert obtained integrals of this type while considering real and imaginary parts of Fourier integral transforms of real functions. Details of these considerations are given in Ref. 15. Tables of Hilbert transforms, as well as various properties which can facilitate the extension of these

tables, appear in Ch.XV of Ref. 14. The reader should note that the PV integral (3) and the formal definition of a Hilbert transform differ by a numerical factor.

It should be noticed that having defined the PV as above, we still have the task of evaluating the two finite integrals inside the brackets. The definition does not provide a method for performing the integration. But we shall see that in certain cases, where additional restrictions are imposed on $f(x,\lambda)$, one is able to find the PV with sufficient ease.

Another remark that concerns this definition, is that in the usual criterion for the existence of I_a^b, namely

$$I_a^b = \lim_{\substack{\eta\to 0\\ \eta>0}} \int_a^{\xi-\eta} \frac{f(x,\lambda)}{x-\xi}\,dx + \lim_{\substack{\rho\to 0\\ \rho>0}} \int_{\xi+\rho}^{b} \frac{f(x,\lambda)}{x-\xi}\,dx \tag{4}$$

η and ρ are required to tend to zero <u>independently</u>. On the other hand, in eq. (3) we have restricted the two limit processes by setting $\eta = \rho = \epsilon$. Hence η, ρ are no longer independent of each other.

The notion of PV may be applied not only when the singularity occurs at a finite distance from the origin, but also when the integrand $F(x,\lambda)$ in the integral $\int_{-\infty}^{\infty} F(x,\lambda)dx$ behaves at $\pm\infty$ in a way that renders the integral divergent.

For this type of integral the definition of PV takes the form

$$P\int_{-\infty}^{\infty} F(x,\lambda)\,dx \equiv \lim_{0<R\to\infty} \int_{-R}^{R} F(x,\lambda)\,dx \tag{5}$$

whereas the ordinary integral of $F(x,\lambda)$ exists if the integral

$$\lim_{\substack{0<R\to\infty\\ 0<\rho\to\infty}} \int_{-\rho}^{R} F(x,\lambda)dx \tag{6}$$

where R and ρ are independent, converges.

In eq. (5), the two independent limit processes of the ordinary definition have been again reduced to a single limit process.

In Appendix I we discuss the evaluation of the PV of integrals with the aid of contour integration in the complex plane for a certain class of integrands.

Finally it should be remarked that if the integrals (2), (4) and (6) exist, then so do the appropriate PV integrals, and the actual values of the respective pairs are identical. The converse of this result is obviously false.

b. Integral Evaluated Along Deformations of the Original Path -- The Plemelj Formulas

It is not possible to ascertain when and who was the first to propose path deformation in order to deal with singularities occurring along the path of integration. It will surely surprise no one that the procedure originated already with Cauchy. After all, Cauchy, together with Riemann, was the founder of the theory of complex functions and complex integration. But the actual results in the present context were presented much later by Plemelj, as described below in detail.

Since the method is widely used in physics (and undoubtedly in engineering as well), at this point it is easier to give as reference a textbook, such as the one by Dennery (Ref. 1).

The present approach consists of two changes in our interpretation of the original integral. First, we look upon the path of integration as being imbedded in the complex plane, and second, we deform the path so as to avoid passing through the singularity. The first change implies that the real variable x is replaced by the complex variable $z = x + iy$. Concerning the subsequent deformation of the path, we shall make the further assumption that the integrand is an analytic function of z. While this is not too restrictive in applications, it allows us to use Cauchy's integral theorem (Ref. 2). In our context, this theorem implies that the path of integration may be continuously deformed inside any region of the z-plane which contains no singularity of the integrand, <u>without</u> changing the value of the integral. If such singularities are encountered during path deformation, the new path must be <u>indented</u> around these in an appropriate fashion.

The procedure is depicted in Figs. 1 and 2. The original path is shown in Fig. 1, and Fig. 2 shows the two possible deformations in the complex z-plane. The indentations shown there are smooth curves, but otherwise quite arbitrary. By Cauchy's theorem, for analytic integrands we may replace these arbitrary indentations by semicircles centered at $x=\xi$. This facilitates the computations without changing the actual values of the corresponding integrals.

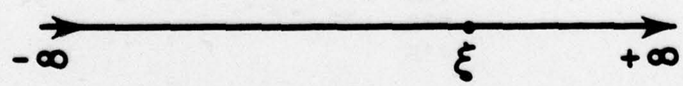

Figure 1. Original Path

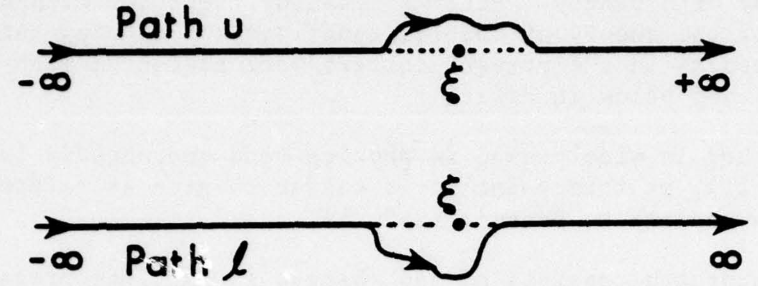

Figure 2. Deformed Paths

In an obvious notation, the two additional values of integral (1), are

$$I_u(\xi,\lambda) = \int_{u} \frac{f(z,\lambda)}{z-\xi}\, dz \tag{7}$$

$$I_\ell(\xi,\lambda) = \int_{\ell} \frac{f(z,\lambda)}{z-\xi}\, dz \tag{8}$$

The actual evaluation of I_u and I_ℓ proceeds as follows:

$$\int_{u} \frac{f(z,\lambda)}{z-\xi}\, dz = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \left[\int_{-\infty}^{\xi-\epsilon} \frac{f(x,\lambda)}{x-\xi}\, dx + \int_{\xi+\epsilon}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx\right]$$

$$+ \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \int_{C_\epsilon^+} \frac{f(z,\lambda)}{z-\xi}\, dz \tag{10}$$

In eq. (10) C_ξ^+ denotes the semicircle centered at $x=\xi$ and with radius ϵ. For I_u to exist it is necessary for the integrals on the r.h.s. of (10) to exist. First we notice that the limit of the square brackets represents the PV of the integral (1) as defined by eqs. (2) and (3). It remains to evaluate the integral along C_ξ^+ and then pass to the limit $\epsilon\to 0$.

On C_ε^+, $z = \xi + \varepsilon e^{i\psi}$, $0 \leq |\psi| \leq \pi$. Hence $dz = i\varepsilon e^{i\psi} d\psi$ and we obtain

$$\int_{C_\varepsilon^+} \frac{f(z,\lambda)}{z-\xi}\, dz = i\varepsilon \int_\pi^0 \frac{e^{i\psi} f(\xi+\varepsilon e^{i\psi},\lambda)}{\varepsilon e^{i\psi}}\, d\psi \tag{11}$$

If $f(x,\lambda)$ is such that integration over ψ may be legitimately interchanged with the limit process $\varepsilon \to 0$, we obtain after obvious simplifications

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon^+} \frac{f(z,\lambda)}{z-\xi}\, dz = -i\pi f(\xi,\lambda) \tag{12}$$

The final result is

$$I_u = \int \frac{f(z,\lambda)}{z-\xi}\, dz = P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi} dx - i\pi f(\xi,\lambda) \tag{13}$$

An entirely similar calculation yields

$$I_\ell = \int \frac{f(z,\lambda)}{z-\xi}\, dz = P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx + i\pi f(\xi,\lambda) \tag{14}$$

The above derivation shows clearly that these two values I_u, I_ℓ were obtained by shrinking the indentation, after the indicated integrations have been performed. Thereby I_u and I_ℓ have this limit process in common with the PV of integral (1). It is therefore not surprising that eqs. (13) and (14) contain, as part of the results, this very same PV integral.

Another aspect that should be highlighted is that if $f(x,\lambda)$ is a <u>real</u> function, then I_u and I_ℓ are <u>complex</u> functions of (ξ,λ). By contrast, the PV, if it exists, is in this case a <u>real</u> function of (ξ,λ).

The results of eqs. (13) and (14) were published first by I. Plemelj in 1908 (Ref. 3), in a more general context. In his investigations the path of integration was from the start a curve on the z-plane, such that $z=\xi$ was on this curve and ξ was not necessarily real.

A different representation may be obtained for I_u and I_ℓ if the function $f(z,\lambda)$ is analytic in z in the neighborhood of the real axis. This is the same requirement that made possible the choice of the semicircular paths C_ε^+ in the derivation of eqs. (13) and (14). In such a case the path is entirely equivalent, by Cauchy's theorem, to a path parallel to the x-axis and at a distance ε above it, which is ultimately let tend to 0. That is, the paths in Fig. 3 are equivalent.

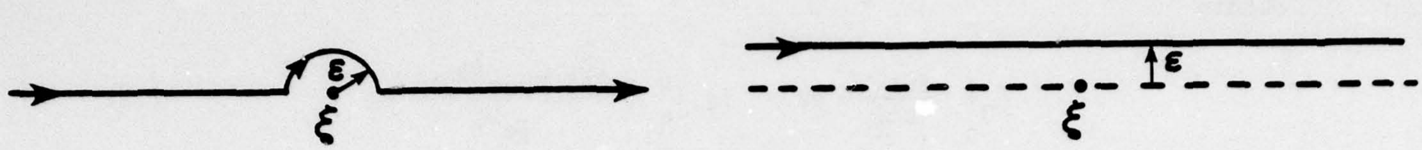

Figure 3.

Hence, we write

$$I_u = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{f(z,\lambda)}{z-\xi}\, dz \tag{15}$$

Now we change the variable of integration as follows:

$$z = \zeta + i\varepsilon \tag{16}$$

Then it is clear that $-\infty \leq \zeta \leq \infty$, and we can write

$$I_u = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{-\infty}^{\infty} \frac{f(\zeta + i\varepsilon,\lambda)}{\zeta - \xi + i\varepsilon}\, d\zeta \tag{17}$$

Since $f(\zeta,\lambda)$ is analytic on the real axis and $f(\zeta + i\varepsilon,\lambda) \to f(\zeta,\lambda)$ as $\varepsilon \to 0$, we may put

$$I_u = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi+i\varepsilon}\, dx \tag{18}$$

In (18) we have replaced the variable of integration ζ by x, only to be reminded of the original integral (1).

A similar result may be obtained for I_ℓ:

$$I_\ell = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi-i\varepsilon}\, dx \tag{19}$$

We close this subsection with another look at I_u, I_ℓ as represented in eqs. (13), (14), this time from a different perspective. Since the function $f(z,\lambda)$, aside from being analytic, is completely arbitrary, we can consider $(x-\xi)^{-1}$ to be a generalized function (Ref. 4). Then the following operational interpretation is immediately obtained:

$$\int_{\frown} \frac{dz}{z-\xi} = P \int_{-\infty}^{\infty} dx \frac{1}{x-\xi} - i\pi \int_{-\infty}^{\infty} dx\ \delta(x-\xi) \tag{20}$$

$$\int_{\smile} \frac{dz}{z-\xi} = P \int_{-\infty}^{\infty} dx \frac{1}{x-\xi} + i\pi \int_{-\infty}^{\infty} dx\ \delta(x-\xi) \tag{21}$$

where $(x-\xi)^{-1}$ and $\delta(x-\xi)$, the Dirac δ-function, act as the kernels of the indicated integral operators.

We shall see in subsection (c) that (20) and (21) are but special cases of a whole class of similar results containing a δ-function.

If we remove the integration signs in the above relations, we obtain the symbolic formulas

$$\frac{1}{x-\xi+i0} = P\ \frac{1}{x-\xi} - i\pi\ \delta(x-\xi) \tag{22}$$

$$\frac{1}{x-\xi-i0} = P\ \frac{1}{x-\xi} + i\pi\ \delta(x-\xi) \tag{23}$$

These results were also derived in Ref. 5, but in quite a different way.

In Appendix I we show how to evaluate the PV through the use of contour integration in the complex plane. First, it is shown there that the simple requirement of analyticity of $f(z,\lambda)$ in the neighborhood of the real axis has to be augmented by more stringent conditions for the contour integration method to be valid. But the evaluation of the PV integral by this method involves essentially the evaluation of the integrals I_u and I_ℓ (by the same method). Thus we see that Plemelj's formulas (13) and (14) remain valid for a larger class of functions $f(z,\lambda)$ than that described in Appendix I. This remark is important in applications, since the conditions of Appendix I may be too restrictive. Examples to this effect are adduced in Appendix II.

c. The δ-Function Interpretation

We return to the original integral (1) and remark that the integrand there is the ratio of two functions, $f(x,\lambda)$ and $x-\xi$. We may look upon this ratio as being the solution of the following equation in g:

$$(x-\xi)\ g = f(x,\lambda) \tag{24}$$

But in writing for g the ratio $f(x,\lambda)/(x-\xi)$ we are neglecting a host of additional values for g, which solve eq. (24). These are obtained by adding to $f(x,\lambda)/(x-\xi)$ all the possible solutions to the homogeneous equation

$$(x-\xi)\ g_o(x,\xi) = 0 \tag{25}$$

At first sight eq. (25) may seem to have only the trivial solution $g_o = 0$. This is indeed correct for all $x \neq \xi$. But for $x = \xi$, we see that g_o can have any arbitrary value $A(\xi)$, where $A(\xi)$ may be a real or complex function.

To find the generalized function interpretation of $g_o(x,\xi)$, we borrow Lighthill's approach for the same problem (Ref. 6). We construct from an arbitrary test function $\psi(x)$ (Ref. 4), another test function $\psi_1(x)$:

$$\psi_1(x) = \psi(x) - \psi(\xi)e^{-x^2} / (x-\xi) \tag{26}$$

Notice that $\psi_1(x)$ has no singularity at $x=\xi$.

Now we wish to show that $g_o(x,\xi)$ has the sifting property of a δ-function, namely

$$\int_{-\infty}^{\infty} g_o(x,\xi)\psi(x)dx = C(\xi)\psi(\xi) \tag{27}$$

where $C(\xi)$ is arbitrary.

Indeed, using eqs. (25) and (26), we find

$$\int_{-\infty}^{\infty} g_o(x,\xi)\psi(x)dx = \int_{-\infty}^{\infty} (x-\xi)\, g_o(x,\xi)\, \psi_1(x)dx + \psi(\xi)\int_{-\infty}^{\infty} g_o(x,\xi)\, e^{-x^2}\, dx$$

$$= \psi(\xi) \int_{-\infty}^{\infty} g_o\, (x,\xi)\, e^{-x^2}\, dx \tag{28}$$

The last integral in eq.(28) is seen to be equal to some arbitrary function $C(\xi)$, since $g_o(x,\xi)$ was considered arbitrary. This proves eq. (27).

If we choose now for $\psi(x)$ the unitary function $U(x) = 1$, for all x, then eq. (28) shows that

$$\int_{-\infty}^{\infty} g_o(x,\xi)dx = C(\xi)$$

All these results imply that we may write for $g_o(x,\xi)$:

$$g_o(x,\xi) = C(\xi)\; \delta(x-\xi) \tag{29}$$

where $\delta(u)$ is the Dirac δ-function (Ref.6).

The result of eq. (29) can be also proven with the aid of Fourier transforms. A discussion of this approach for a more general case is given in Section III(c).

It is clear now that

$$g(x) = \frac{f(x,\lambda)}{x-\xi} + C\,\delta(x-\xi) \tag{30}$$

is the general solution of eq. (24), for all x belonging to an interval which contains ξ in its interior. If ξ is not inside this interval, then in eq. (30) $C\,\delta(x-\xi)$ must be deleted. We also require $f(\xi,\lambda) \neq 0$.

We can summarize the preceding discussion with the statement: division by zero is legitimate if the infinity introduced thereby is taken explicitly into account. Having said this implies that in eq. (30) we should look upon the ratio $f(x,\lambda)/(x-\xi)$ as being finite, <u>even</u> at $x = \xi$. To accomplish this we recall that C is arbitrary and it will still be so if multiplied by $f(\xi,\lambda)$. Then we can write instead of (30)

$$g(x) = \left\{\frac{1}{x-\xi} + C\,\delta(x-\xi)\right\} f(x,\lambda) \tag{31}$$

If we let $f(x,\lambda) \equiv 1$, the result is identical with the one given in Ref. 4.

Now, since our purpose is to use $(x-\xi)^{-1}$ under the integral sign, we shall choose for $(x-\xi)^{-1}$ on the r.h.s. of (31) a particular determination that yields a finite value. In most cases of interest, but not all, the choice is the PV of the integral.

Hence all possible determinations of $g(x) \equiv (x-\xi)^{-1}$ can be written symbolically as

$$\left.\frac{1}{x-\xi}\right|_g = \left.\frac{1}{x-\xi}\right|_p + C\,\delta(x-\xi) \tag{32}$$

where subscript g and p denote general and particular solutions, respectively.

In integral form we have:

$$\int \frac{f(x,\lambda)}{x-\xi}\,dx = P\int \frac{f(x,\lambda)}{x-\xi}\,dx + Cf(\xi,\lambda) \tag{33}$$

where we have taken the PV for p.

Eq. (32) includes as particular cases eqs. (22) and (23), while eq. (33) includes as particular cases I_u and I_ℓ of eqs. (13) and (14).

It is of interest to note that Jones (Ref. 4) defines, in common with Lighthill, $(x-\xi)^{-1}$ as that generalized function which selects the PV when appearing as a kernel of an integral. Obviously, this is a special case of the general result in eq. (33).

III. INTEGRALS WITH MULTIPLE POLES

Here we shall consider integrals of the type

$$I\,(\xi,\lambda) = \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\,dx \tag{34}$$

The function $f(x,\lambda)$ has the same properties as in Section II, and $n > 1$ is an integer.

a. Principal Value Interpretation

If we wish to follow a pattern similar to the one of section II we encounter a difficulty in attempting to define the PV of (34). To clarify the situation we take the special case $f(x,\lambda) \equiv 1$. With the usual definition of PV, we obtain by direct calculation for $n > 1$

$$P \int_{-\infty}^{\infty} \frac{dx}{(x-\xi)^n} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1+(-1)^n}{(n-1)\varepsilon^{n-1}} = \begin{cases} 0; & n \text{ odd} \\ \infty; & n \text{ even} \end{cases} \tag{35}$$

For odd n the above result corresponds to our expectation, since the area under an antisymmetric (with respect to ξ) curve should be zero. On the other hand, we still have a divergent integral for n even. One way to remedy this is to redefine the PV as follows

$$P_{\text{redef}} \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\,dx \equiv \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \varepsilon^n \left[\int_{-\infty}^{\xi-\varepsilon} \frac{f(x,\lambda)}{(x-\xi)^n}\,dx + \int_{\xi+\varepsilon}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\,dx\right] \tag{36}$$

For $f(x,\lambda) \equiv 1$, we obtain

$$P_{\text{redef}} \int_{-\infty}^{\infty} \frac{dx}{(x-\xi)^n} = 0 \text{ for all integral } n > 1 \tag{37}$$

In Appendix II we show that $P \int_{-\infty}^{\infty} dx/(x-\xi) = 0$. It is clear from (36) that we have also

$$P_{\text{redef}} \int_{-\infty}^{\infty} \frac{dx}{x-\xi} = 0 \tag{38}$$

Hence (37) is true for all integral $n > 0$.

Whether this redefined PV is a useful device or a more general concept has to be introduced, is still an open question. In the meantime, another possibility exists: we start with eq. (34) and integrate by parts n-1

times. We assume that at $x = \pm\infty$, the function $f(x,\lambda)$ and its first n-1 derivatives with respect to x do not grow faster than certain powers of x, such that $f^{(k)}(x,\lambda)/x^{n-1-k} \to 0$, for $0 \leq k \leq n-2$.

Then we find

$$\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\, dx \tag{39}$$

Using this relation, when the conditions of its validity are satisfied, we define the PV of (34) as

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx \equiv \frac{1}{(n-1)!}\, P \int_{-\infty}^{\infty} \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\, dx$$

$$= \frac{1}{(n-1)!} \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \left[\int_{-\infty}^{\xi-\epsilon} \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\, dx + \int_{\xi+\epsilon}^{\infty} \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\, dx \right] \tag{40}$$

We have used script P to denote this more general type of PV.

We note that with this definition, if $f(x,\lambda) \equiv 1$ for $-\infty \leq x \leq \infty$, we obtain the same result as in eq. (37).

In eq. (39) we have assumed that all the derivatives concerned exist. If we interpret the function $f(x,\lambda)$ to be a generalized function, such as defined in Refs. (4) and (5), these derivatives always exist and are generalized functions themselves. When we replace $f(x,\lambda)$ in (39) and (40) by $[H(x-a) - H(x-b)]\, f(x,\lambda)$ we obtain the result given by Jones in § 4.4 of Ref. 4, since during the integration by parts procedure, the integrated terms no longer vanish. In the same reference one finds the connection, for this case, with the "Finite Part" concept introduced by Hadamard (Ref. 11). Jones (Ref. 4) seems to have been anticipated by Fox (Ref. 12), who used Hadamard's ideas to generalize the PV of an integral of type (34), along a finite interval. Fox obtains basically the same result as eq. (40) for the infinite interval, as well as the generalization of Plemelj formulas given below. It appears that the derivations of Fox, based on standard analytic arguments, apply to a more restricted class of functions than the interpretation in terms of generalized functions given by Jones.

b. Plemelj's Generalized Formulas

By analogy with the treatment for n = 1, we define two integrals along deformed paths:

$$I_u(\xi,\lambda) \equiv \int_{\frown} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx \tag{41}$$

$$I_{\ell}(\xi,\lambda) \equiv \int_{\smile} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx \tag{42}$$

The paths in (41) and (42) are those shown in Fig. 2. The difficulty encountered in (a) above is present here too when attempting to evaluate (41) and (42) according to the prescription given in eq. (10). This becomes clear when we write down the equation corresponding to eq. (11):

$$\int_{C_\varepsilon^+} \frac{f(z,\lambda)}{(z-\xi)^n}\, dz = \frac{i}{\varepsilon^{n-1}} \int_{\pi}^{0} e^{-i(n-1)\psi} f(\xi+\varepsilon e^{i\psi},\lambda)\, d\psi \tag{43}$$

It is seen immediately that when $\varepsilon \to 0$ the r.h.s. of eq. (43) diverges. Moveover, the contribution to I_u from the path along the real axis also diverges, as we have seen in (a).

Therefore we proceed here as in (39) and first write

$$\int_{\frown} \frac{f(z,\lambda)}{(z-\xi)^n}\, dz = \frac{1}{(n-1)!} \int_{\frown} \frac{f^{(n-1)}(z,\lambda)}{z-\xi}\, dz \tag{44}$$

On using the prescription of eq. (10) for the r.h.s. of eq. (44), we obtain

$$I_u = \int_{\frown} \frac{f(z,\lambda)}{(z-\xi)^n}\, dz = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx - \frac{i\pi f^{(n-1)}(\xi,\lambda)}{(n-1)!} \tag{45}$$

In a similar fashion the analogue of eq. (14) is

$$I_{\ell} = \int_{\smile} \frac{f(z,\lambda)}{(z-\xi)^n}\, dz = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{(x-\xi)^n}\, dx + \frac{i\pi f^{(n-1)}(\xi,\lambda)}{(n-1)!} \tag{46}$$

We note that (45) and (46) reduce to Plemelj's formulas (13) and (14) when $n = 1$. Hence we call (45) and (46) Plemelj's Generalized Formulas.

The following results correspond to the results in eqs. (18), (19), (20), (21), (22) and (23):

$$I_u = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{f^{(n-1)}(x,\lambda)}{x-\xi+i\varepsilon}\, dx \tag{47}$$

$$I_{\ell} = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{f^{(n-1)}(x,\lambda)}{x-\xi-i\varepsilon}\, dx \tag{48}$$

$$\int_{\frown\!\to} \frac{dz}{(z-\xi)^n} = \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{(x-\xi)^n} - \frac{i\pi(-1)^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} dx\delta^{(n-1)}(x-\xi) \tag{49}$$

$$\int_{\smile\!\to} \frac{dz}{(z-\xi)^n} = \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{(x-\xi)^n} + \frac{i\pi(-1)^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} dx\delta^{(n-1)}(x-\xi) \tag{50}$$

$$\frac{1}{(x-\xi+i0)^n} = \mathcal{P}\frac{1}{(x-\xi)^n} - \frac{i\pi(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x-\xi) \tag{51}$$

$$\frac{1}{(x-\xi+i0)^n} = \mathcal{P}\frac{1}{(x-\xi)^n} + \frac{i\pi(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x-\xi) \tag{52}$$

c. The δ-Function Interpretation

In this subsection we are concerned with the general solution to the equation

$$(x-\xi)^n g(x) = 1 \tag{53}$$

For $n = 1$, we have found

$$g_1(x) = \frac{1}{x-\xi}\Big|_p + A_1\,\delta(x-\xi) \tag{54}$$

where the subscript p denotes a particular interpretation of $(x-\xi)^{-1}$, which in eq. (32) we took to be P, the Principal Value.

We proceed by induction. Having solved for $g_1(x)$ from the equation

$$(x-\xi)g_1(x) = 1 \tag{55}$$

we solve now for $g_2(x)$ from

$$(x-\xi)^2 g_2(x) = 1 \tag{56}$$

But we can write (56), on using (55), as

$$(x-\xi)g_2(x) = g_1(x) \tag{57}$$

Use of (54) on both sides of (57) yields,

$$g_2(x) = \frac{1}{(x-\xi)^2}\Big|_p + A_1\frac{\delta(x-\xi)}{x-\xi} + A_2\,\delta(x-\xi) \tag{58}$$

To bring (58) to the final form, we prove the following result concerning the δ-function and its derivatives:

$$\delta^{(r)}(x) = -x\delta^{(r+1)}(x) \qquad (59)$$

for all integral $0 \leq r$.

Multiplying (59) by an admissible function and integrating by parts its r.h.s., it is easily proved that (59) is indeed valid.

From (59) we obtain

$$\frac{\delta^{(r)}(x-\xi)}{x-\xi} = -\delta^{(r+1)}(x-\xi) \qquad (60)$$

and in particular for $r = 0$, we have

$$\frac{\delta(x-\xi)}{x-\xi} = -\delta'(x-\xi) \qquad (61)$$

Hence we can write for (58)

$$g_2(x) \equiv \left.\frac{1}{(x-\xi)^2}\right|_g = \left.\frac{1}{(x-\xi)^2}\right|_p + C_o\delta(x-\xi)+C_1\delta'(x-\xi) \qquad (62)$$

where C_0, C_1 are arbitrary constants.

It is clear now that proceeding in this way and using eq. (60) repeatedly we find for arbitrary integral $n > 0$

$$\left.\frac{1}{(x-\xi)^n}\right|_g = \left.\frac{1}{(x-\xi)^n}\right|_p + \sum_{r=0}^{n-1} C_r\,\delta^{(r)}(x-\xi) \qquad (63)$$

Without specifying the particular value p, this result is given by Jones in eq. § 6.2 of Ref. 4.

In eq. (63) we choose for p the PV as defined in eq. (40). Hence we can finally write the symbolic formula

$$\left.\frac{1}{(x-\xi)^n}\right|_g = \mathcal{P}\frac{1}{(x-\xi)^n} + \sum_{r=0}^{n-1} C_r\,\delta^{(r)}(x-\xi) \qquad (64)$$

Using this result we can write the analogue of eq. (33)

$$\int_{-\infty}^{\infty}\frac{f(x,\lambda)}{(x-\xi)^n}\,dx = \mathcal{P}\int_{-\infty}^{\infty}\frac{f(x,\lambda)}{(x-\xi)^n}\,dx + \sum_{r=0}^{n-i} K_r\,f^{(r)}(\xi,\lambda) \qquad (65)$$

The general solution of eq. (53) presented in eq. (64) can be obtained also by applying a Fourier transform to eq. (53), solving the resulting linear differential equation in G(k) (the transform of g(x)), and then performing the inverse Fourier transform which yields g(x). The arbitrary constants C_r make their appearance as the constants of integration of the differential equation. Since the differential equation is inhomogeneous (i.e., it has a non zero term on the r.h.s.) its solution consists of the sum of a particular solution and the general solution to the homogeneous equation. This is precisely the result presented in eq. (63).

IV. HALF-RANGE INTEGRALS

In this section we consider integrals of the type

$$I(\lambda,\xi) = \int_0^\infty \frac{f(x,\lambda)}{x-\xi}\,dx \tag{66}$$

where $\xi > 0$. This is representative of the half-range integrals along the negative real axis as well, since a change of variable will bring the integral to the form (66), provided, of course, that $f(x,\lambda)$ has the required properties. Furthermore, integrals of the type

$$I(\lambda,\xi,a) = \int_a^\infty \frac{f(x,\lambda)}{x-\xi}\,dx \tag{67}$$

where $\xi > a \geq 0$, are expressible in the form (66) by a translation

$$I(\lambda,\xi,a) = I(\lambda,\xi-a) = \int_0^\infty \frac{f(x+a,\lambda)}{x-(\xi-a)}\,dx \tag{68}$$

where, by the previous conditions, we have $\xi - a > 0$.

It is not difficult to show that Plemelj formulas (13), (14) are applicable here as well, such that

$$I_u \equiv \int \frac{f(x,\lambda)}{z-\xi}\,dz = P\int_0^\infty \frac{f(x,\lambda)}{x-\xi}\,dx - i\pi f(\xi,\lambda) \tag{69}$$

$$I_l \equiv \int \frac{f(z,\lambda)}{z-\xi}\,dz = P\int_0^\infty \frac{f(x,\lambda)}{x-\xi}\,dx + i\pi f(\xi,\lambda) \tag{70}$$

The same is true of the δ-function interpretation as discussed in II(c).

Therefore it remains to evaluate the PV of integral (66), by any method. First, we note the following

$$P\int_0^\infty \frac{dx}{x-\xi} = \infty, \quad \xi>0 \tag{71}$$

Hence, unlike the full-range integral, the PV of (66), for $f(x,\lambda) \equiv 1$, diverges. This result immediately precludes the evaluation of the PV by subtraction of the singular part as shown in (I-17) for the full-range integrals. Also, writing $f(x,\lambda)$ as a sum of an odd and an even function does not yield any advantage in the present case. Nevertheless, several possibilities do exist for the PV evaluation of (66), depending on the nature of the function $f(x,\lambda)$.

A finite particular value to be associated with the integral $\int_0^\infty dx/(x-\xi)$ is offered in Appendix VI, in the spirit of the Finite Part of Hadamard (Ref. 11) and the δ-function interpretation of Section II(c).

a. <u>PV of Full-Range Integral is Known</u>

If we assume that $f(x,\lambda)$ is defined over the full-range $|x| \leq \infty$, and that the PV can be explicitly evaluated, i.e.,

$$P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi} dx = F(\xi,\lambda) \tag{72}$$

then, if $\xi > 0$, we can write

$$P \int_0^\infty \frac{f(x,\lambda)}{x-\xi} dx = F(\xi,\lambda) + \int_0^\infty \frac{f(-x,\lambda)}{x+\xi} dx \tag{73}$$

The integral on the r.h.s. of (73) is an ordinary integral now (the integrand has no singularities) and moreover it is a convergent integral, since $F(\xi,\lambda)$ is finite by assumption. Thus, formula (73) allows us to dispense with the ϵ limit in the PV definition. This approach may be practical, since the form of the integral on the r.h.s. of (73) shows it to be the Stieltjes transform of $f(-x,\lambda)$. Tables of such integrals are given in Ref. (14).

b. <u>Evaluation by Contour Integration</u>

Let us assume that the appropriate requirements are satisfied by $f(x,\lambda)$ when extended to the sector of the complex plane given by $|\arg z| \leq \alpha$, $0 < \alpha \leq \pi/2$.

To evaluate the PV of (66) we basically evaluate I_u and I_ℓ, in a fashion entirely analogous to the one described in Appendix I. But here the results are not the same, as it will become clear from the typical case discussed below.

First, we evaluate I_u, by choosing the contours in Figs. 4 and 5, each contour corresponding to the specific behavior of $f(z,\lambda)$.

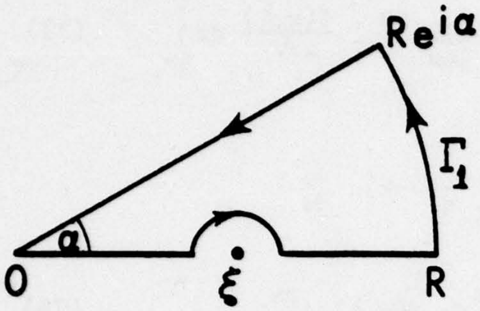

Fig. 4

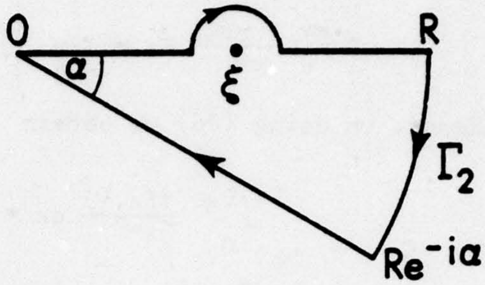

Fig. 5

Thus for $\lambda > \lambda_0$ the contour of Fig. 4 may be the appropriate one. We assume that the integral along Γ_1 vanishes in the limit $R \to \infty$, and the final result is

$$P \int_0^\infty \frac{f(x,\lambda)}{x-\xi}\, dx = i\pi f(\xi,\lambda) + \int_0^\infty \frac{f(e^{i\alpha}y,\lambda)}{y-\xi e^{-i\alpha}}\, dy$$

$$+2\pi i \sum_{0<\arg z<\alpha} \mathrm{Res}\, \frac{f(z,\lambda)}{z-\xi};\ \lambda<\lambda_0 \tag{74}$$

Similarly, for $\lambda < \lambda_0$, the contour of Fig. 5 may be the appropriate one, and with the integral along Γ_2 vanishing with $R \to \infty$, we find

$$P \int_0^\infty \frac{f(x,\lambda)}{x-\xi}\, dx = i\pi f(\xi,\lambda) + \int_0^\infty \frac{f(e^{-i\alpha}y,\lambda)}{y-\xi e^{i\alpha}}\, dy$$

$$-2\pi i \sum_{-\alpha\leq\arg z\leq 0} \mathrm{Res}\, \frac{f(z,\lambda)}{z-\xi};\ \lambda<\lambda_0 \tag{75}$$

We observe immediately that the integrals on the r.h.s. of (74) and (75) are ordinary integrals, devoid of singularities along the path of integration (provided α is so chosen that $f(z,\lambda)$ has no singularities along the rays $z = e^{\pm i\alpha}y$; $0 < y$).

The evaluation of I_ℓ proceeds in exactly the same way and we therefore omit it.

c. <u>Evaluation by Laplace Transform Identity</u>

We make use in this section of the identity

$$\int_0^\infty e^{-au}\, du = \frac{1}{a};\ \mathrm{Re}\ a > 0 \tag{76}$$

This is none other than the Laplace transform of the Heaviside function H(u).

Now we can write from the definition of PV and of an integral with an infinite upper limit:

$$P\int_0^\infty \frac{f(x,\lambda)}{x-\xi}\,dx = \lim_{\epsilon\to 0}\left\{\int_0^{\xi-\epsilon} \frac{f(x,\lambda)}{x-\xi}\,dx + \lim_{R\to\infty}\int_{\xi+\epsilon}^{R} \frac{f(x,\lambda)}{x-\xi}\,dx\right\} \tag{77}$$

Hence, on using (76) we obtain

$$\int_0^{\xi-\epsilon} \frac{f(x,\lambda)}{x-\xi}\,dx = -\int_0^{\xi-\epsilon} dx f(x,\lambda)\int_0^\infty e^{-(\xi-x)u}\,du$$

$$= -\int_0^\infty du\, e^{-\xi u}\int_0^{\xi-\epsilon} dx\, f(x,\lambda)\, e^{ux} \tag{78}$$

where we have assumed that the interchange of integrations is permissible.

Similarly we find

$$\int_{\xi+\epsilon}^{R} \frac{f(x,\lambda)}{x-\xi}\,dx = \int_0^\infty du\, e^{u\xi}\int_{\xi+\epsilon}^{R} dx\, f(x,\lambda) e^{-ux} \tag{79}$$

On adding (78) and (79) and taking the limits indicated in (77), we find

$$P\int_0^\infty \frac{f(x,\lambda)}{x-\xi}\,dx = \int_0^\infty du\,\left|e^{u\xi}\int_\xi^\infty dx\, f(x,\lambda)e^{-ux} - e^{-u\xi}\int_0^\xi dx\, f(x,\lambda)\, e^{ux}\right| \tag{80}$$

The integrals appearing in (80) are ordinary integrals (the integrands have no singularities along the paths of integration) which converge if $f(x,\lambda)$ has the appropriate behavior as $x \to \infty$.

We remark that the present method may work when methods (a) and (b) will fail. An example of such behavior arises for $f(x,\lambda) = e^{-\lambda x}$, $\lambda > 0$, which will be discussed in Appendix IV.

Incidentally, method (c) may be advantageous in evaluating the PV of full-range integrals, although we have not used it for this purpose in the present report.

We consider now the case of higher order poles. First, it is easily seen from the definition that

$$P\int_0^\infty \frac{dx}{(x-\xi)^n} = \begin{cases} \infty & ;\ n \text{ even} \\ \dfrac{(-1)^{n-1}}{(n-1)\xi^{n-1}} & ;\ n \text{ odd} \end{cases} \tag{81}$$

where $n > 1$.

Hence, as in the case of the full-range integral, we must generalize the definition of PV in the same way that was done in Section III.

The definition is

$$\mathcal{P}\int_0^\infty \frac{f(x,\lambda)}{(x-\xi)^n}\,dx = \frac{1}{(n-1)!}\,P\int_0^\infty \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\,dx \tag{82}$$

We can now make use of all the methods known, such as (a), (b) and (c) above, to evaluate the r.h.s. of (82).

The Plemelj formulas may be obtained now by replacing the PV integral on the r.h.s. of (82) with the appropriate expressions, as is done in (45) and (46).

V. INTEGRALS WITH FINITE LIMITS

The integrals we discuss here are of the type

$$I(\xi,\lambda) = \int_a^b \frac{f(x,\lambda)}{x-\xi}\,dx \tag{83}$$

where $a < \xi < b$.

The PV is defined as in eq. (3). This PV can be considered as the finite Hilbert transform (up to a numerical factor). A few references to this type of PV integral may be found in Ch. XV of Ref. 14. The actual evaluation of the PV is hampered by our inability to find the anti-derivative of the integrand. Therefore, we again must resort to other methods in order to avoid the ε-limit process.

But if the PV of (83) exists, then the Plemelj integrals I_u and I_ℓ also exist:

$$I_u \equiv \int \frac{f(z,\lambda)}{z-\xi}\,dz = P\int_a^b \frac{f(x,\lambda)}{x-\xi}\,dx - i\pi f(\xi,\lambda) \tag{84}$$

$$I_\ell \equiv \int \frac{f(z,\lambda)}{z-\xi}\,dz = P\int_a^b \frac{f(x,\lambda)}{x-\xi}\,dx + i\pi f(\xi,\lambda) \tag{85}$$

The δ-function interpretation is clearly still valid in the present case also, and the pertinent formulas carry over, with minor changes, from II(c).

Using definition (3), we find immediately

$$P\int_a^b \frac{dx}{x-\xi} = \log\frac{b-\xi}{\xi-a}\ ;\ a<\xi<b \tag{86}$$

We list now several methods of evaluating the PV of (83).

a. <u>Subtraction of Singular Part</u>

Subtraction and addition of the singular part yields

$$P \int_a^b \frac{f(x,\lambda)}{x-\xi}\, dx = \int_a^b \frac{f(x,\lambda) - f(\xi,\lambda)}{x-\xi}\, dx + f(\xi,\lambda)\, P \int_a^b \frac{dx}{x-\xi}$$

$$= \int_a^b \frac{f(x,\lambda) - f(\xi,\lambda)}{x-\xi}\, dx + f(\xi,\lambda) \log \frac{b-\xi}{\xi-a} \tag{87}$$

The integral on the r.h.s. of (87) is an ordinary integral (no singularity at $x = \xi$, if $f'(\xi,\lambda)$ is finite).

b. <u>Contour Integration</u>

The method is used to evaluate the Plemelj integrals I_u, I_ℓ. Here the behavior of $f(z,\lambda)$ as $|z| \to \infty$ is unimportant, since a,b are finite. Thus the choice of contour will be dictated solely by convenience of calculation. One such contour is shown below for the evaluation of I_u.

We choose a half circle Γ in the upper z plane with radius (b-a)/2 and center at x = (b+a)/2. Then we obtain

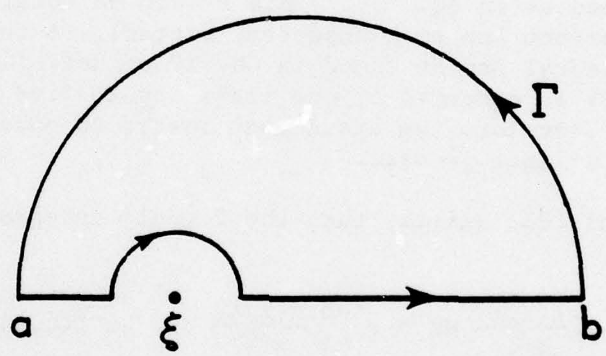

Fig. 6

$$P \int_a^b \frac{f(x,\lambda)}{x-\xi}\, dx = i\pi f(\xi,\lambda) + 2\pi i \sum_{\left(\substack{z \text{ inside} \\ \text{contour}}\right)} \text{Res}\, \frac{f(z,\lambda)}{z-\xi}$$

$$+\, i \int_0^\pi \frac{f\left(\frac{b+a}{2} + \frac{b-a}{2} e^{i\theta}, \lambda\right)}{1 + \frac{b+a-2\xi}{b-a}\, \varepsilon^{-i\theta}}\, d\theta \tag{88}$$

The integral appearing in the r.h.s. of (88) is again an ordinary convergent integral. It is not difficult to verify that the expression obtained when multiplying the denominator by its complex conjugate has no zeros on the real axis.

c. Evaluation by Laplace Transform Identity

Use of (76) in the present case yields

$$P \int_a^b \frac{f(x,\lambda)}{x-\xi}\, dx = \int_0^\infty du \left| e^{u\xi} \int_\xi^b dx\, f(x,\lambda) e^{-ux} - e^{-u\xi} \int_a^\xi dx f(x,\lambda) e^{ux} \right| \tag{89}$$

We notice that the inner integrals are independent of the behavior of $f(x,\lambda)$ at infinity, but the convergence of the integral over u may depend on $f(x,\lambda)$. For the convergent case, this formulation allows us again to dispense with the ε-limit process.

d. Evaluation from PV of Full-Range Integral

This method will apply when the function $f(x,\lambda)$ can be extended along the entire real axis and the PV of the full-range integral exists. Then we can write

$$P \int_a^b \frac{f(x,\lambda)}{x-\xi}\, dx = P \int_{-\infty}^\infty \frac{f(x,\lambda)}{x-\xi}\, dx - \int_{-\infty}^a \frac{f(x,\lambda)}{x-\xi}\, dx - \int_b^\infty \frac{f(x,\lambda)}{x-\xi}\, dx \tag{90}$$

Since we have assumed $a < \xi < b$, the last two integrals are ordinary integrals which moreover converge, since the PV integral over the whole real axis is assumed to exist.

To treat poles of higher order, we follow the procedure mentioned in Section III(a). The resulting formula, after repeated integration by parts, is more complicated since we have to keep track of the contributions from the endpoints of the interval of integration.

The result is therefore the generalized form

$$\mathcal{P} \int_a^b \frac{f(x,\lambda)}{(x-\xi)^n}\, dx = \frac{1}{(n-1)!} P \int_a^b \frac{f^{(n-1)}(x,\lambda)}{x-\xi}\, dx$$

$$- \frac{1}{(n-1)!} \sum_{k=0}^{n-2} (n-2-k)! \left. \frac{f^{(k)}(x,\lambda)}{(x-\xi)^{n-1-k}} \right|_a^b \tag{91}$$

The Plemelj integrals I_u, I_ℓ for this case are now obtained in the same way as in (45) and (46), by replacing the PV integral on the r.h.s. of (91) with the appropriate expressions.

VI. CONCLUSION

The present report has shown, in detail, how to interpret integrals with poles located on the path of integration. In particular, through the use of concepts from the theory of generalized functions, it was established that such integrals possess an infinite set of values.

The choice of a value, to be considered the "correct" value, depends entirely on the context from within which the singular integral arose. The ability of making such a choice depends in turn on knowing the set of values associated with a particular integral. It was shown that knowledge of the Principal Value, when it exists, is equivalent to knowledge of the entire set of values. Thus, several practical methods, which replace the limit process appearing in the definition of the Principal Value, have been presented for the various cases under consideration (full and half-range, and finite intervals).

Several examples were worked out in detail, to illustrate the theory and to exhibit the usefulness as well as the limits of applicability of the various methods of actual evaluation.

A feature, of both practical and mathematical interest, is that the uniqueness of the Principal Value yields integral identities. Such identities enable us to substitute a rapidly convergent integral for a slowly convergent one. The advantages, numerical and otherwise, of such a substitution are well known.

The main purpose of the work reported here was to gather all the known results on integrals with singularities, fill the gaps where necessary, and present the whole subject in a coherent and, hopefully, a usable form. As the work advanced, it became clear that certain omissions will be necessary in order to keep the report at a reasonable length. Several important omissions that deserve a separate treatment are: (i) integrals with branch point singularities, (ii) integral with a singularity (pole or branch point) occuring at one endpoint of the interval of integration, and (iii) singular integrals in spaces of higher dimension. The author hopes to return to these subjects in future reports.

A final remark concerns the theory of singular integral equations where PV integrals of the type discussed here make their appearance. The function $f(x,\lambda)$ in this case is not given and must be found as a solution of the integral equation. Fundamental work in this field is due to S. G. Mikhlin, whose contributions led later on to the theory of pseudo-differential operators, a subject of current interest. Ref. 16 contains an appraisal of this work and extensive references.

APPENDIX I

EVALUATION OF THE PRINCIPAL VALUE

The PV of an integral over a finite interval a,b was defined in eq. (3). When we take the additional limits $a \to -\infty$, $b \to +\infty$ independently, the integral may or may not converge. In both cases we shall denote by Principal Value of a singular integral the expression obtained from the two limit processes

$$P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx = \lim_{R\to\infty} \lim_{\substack{\varepsilon\to 0 \\ \varepsilon>0}} \left[\int_{-R}^{\xi-\varepsilon} \frac{f(x,\lambda)}{x-\xi}\, dx + \int_{\xi+\varepsilon}^{R} \frac{f(x,\lambda)}{x-\xi}\, dx \right] \quad \text{(I-1)}$$

taken in the indicated order. Thus (I-1) may exist when, by the usual definition, the integral diverges.

The evaluation of the PV requires knowledge of the primitive of $f(x,\lambda)/(x-\xi)$. But it is well-known that no general method exists for determining the anti-derivative of an arbitrary function. Hence the analytic evaluation of the PV is no easy task. Fortunately, for many cases of interest in applications, the function $f(x,\lambda)$, when extended to the complex z-plane, has properties that enable us to obtain the PV as a by-product of contour integration.

The method is best explained when we refer to Fig. 7 below. First, we assume that $f(z,\lambda)$ is analytic in the upper half of the z-plane. Then we take the integral around the closed path shown in Fig. 7. If $f(z,\lambda)$ has singularities inside this contour, such as a number of simple poles z_k, then Cauchy's theorem yields

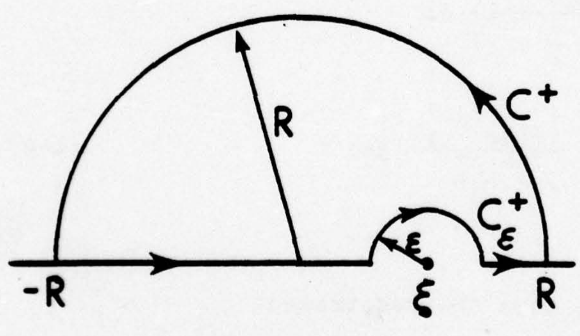

Figure 7

$$\int_{-R}^{\xi-\varepsilon} \frac{f(x,\lambda)}{x-\xi}\, dx + \int_{\xi+\varepsilon}^{R} \frac{f(x,\lambda)}{x-\xi}\, dx + \int_{C_\varepsilon^+} \frac{f(z,\lambda)}{z-\xi}\, dz$$

$$+ \int_{C^+} \frac{f(z,\lambda)}{z-\xi}\, dz = 2\pi i \sum_k \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_k} \quad \text{(I-2)}$$

This equation is valid as it stands, the only requirement being the analyticity of $f(z,\lambda)$, as imposed above. For this result to be useful in the evaluation of the PV, the function $f(z,\lambda)$ must satisfy two additional requirements: (i) $f(\xi,\lambda)$ is finite; (ii) the integral along the semicircle C^+ must vanish in the limit $R \to \infty$.

Note that we have already demanded that (i) be satisfied in our initial discussion of the Principal Value.

To see what requirement (ii) entails, we write for the integral along C^+

$$\int_{C^+} \frac{f(z,\lambda)}{z-\xi}\, dz = iR \int_0^\pi \frac{f(Re^{i\theta},\lambda)}{Re^{i\theta}-\xi}\, e^{i\theta} d\theta$$

$$= i \int_0^\pi \frac{f(Re^{i\theta},\lambda)}{1 - \frac{\xi}{R}\, e^{-i\theta}}\, d\theta \tag{I-3}$$

Hence

$$\left| \int_{C^+} \frac{f(z,\lambda)}{z-\xi}\, dz \right| = \left| \int_0^\pi \frac{f(Re^{i\theta},\lambda)}{1 - \frac{\xi}{R}\, e^{-i\theta}}\, d\theta \right| \leq$$

$$\leq \int_0^\pi \frac{|f(Re^{i\theta},\lambda)|}{\left|1 - \frac{\xi}{R}\, e^{-i\theta}\right|}\, d\theta$$

$$\leq \frac{1}{\sqrt{2}} \int_0^\pi \frac{|f(Re^{i\theta},\lambda)|}{\sqrt{1-\frac{|\xi|}{R}}}\, d\theta \tag{I-4}$$

It is clear from (I-4) that (ii) implies then the requirement

(ii-a) $|f(z,\lambda)| \to 0$ as $|z| \to \infty$, uniformly in arg z, for $0 \leq \arg z = \theta \leq \pi$.

This condition is the modified form of a requirement placed on $f(z,\lambda)$, in a slightly different context, by Whittaker and Watson (Ref. 7 -- § 6.22). A particular case of (ii-a) is a function of the form $f(x,\lambda) = e^{i\lambda x}g(x)$.

Then if $g(z)$ is analytic and $\lambda > 0$, the condition (ii-a) on f translates into: $|g(z)| \to$ constant as $|z| \to \infty$, uniformly in arg z, for $0 \leq \arg z \leq \pi$. This is a modified form of Jordan's lemma (Ref. 7 - § 6.222).

We finally obtain, when taking the limits (indicated in eq. (I-1)) in eq. (I-2)

$$P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi} dx - i\pi f(\xi,\lambda) = 2\pi i \sum_k \text{Res} \left. \frac{f(z,\lambda)}{z-\xi} \right|_{z=z_k} \tag{I-5}$$

The second term on the l.h.s. of (I-5) is the contribution of the integral along C_ξ^+ (when $\epsilon \to 0$), while the sum is extended over the residues of all poles in the upper half plane.

Since the PV integral, if it exits, is unique, its value should not depend on the manner of its evaluation. Hence, if the semicircle of radius ϵ is taken below the real axis, we should obtain, all other things being equal, the same value as in eq. (5).

From Fig. 8 we can write down

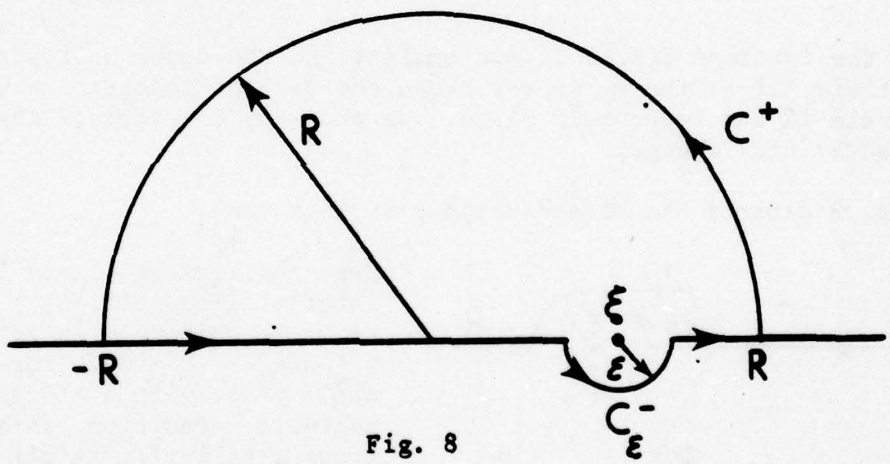

Fig. 8

$$\int_{-R}^{\xi-\epsilon} \frac{f(x,\lambda)}{x-\xi} dx + \int_{\xi+\epsilon}^{R} \frac{f(x,\lambda)}{x-\xi} dx + \int_{C_e^-} \frac{f(z,\lambda)}{z-\xi} dx + \int_{C^+} \frac{f(z,\lambda)}{z-\xi} dz =$$

$$= 2\pi i \left[f(\xi,\lambda) + \sum_k \text{Res} \left. \frac{f(z,\lambda)}{z-\xi} \right|_{z=z_k} \right] \tag{I-6}$$

The appearance of $f(\xi,\lambda)$ on the r.h.s. of (I-6) is due to the fact that the closed contour includes now the pole at $z = \xi$. On the other hand we have

$$\int_{C_\varepsilon^-} \frac{f(z,\lambda)}{z-\xi}\, dz = i \int_0^\pi d\psi\; f(\xi+\varepsilon e^{i\psi},\lambda)$$

Therefore, when $\varepsilon \to 0$ we obtain

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{C_\varepsilon^-} = i\pi f(\xi,\lambda)$$

Collecting all these results, we find

$$P\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx + i\pi f(\xi,\lambda) = 2\pi i \left[f(\xi,\lambda) + \sum_k \frac{f(z,\lambda)}{z-\xi}\bigg|_{z=z_k} \right] \tag{I-7}$$

If the function $f(z,\lambda)$ is not analytic in the upper half plane, or does not satisfy (ii-a) there, we may close the path of integration with a semicircle in the lower half plane. We show how the form of the result is affected by this change.

Fig. 9 depicts the closed contour in this case.

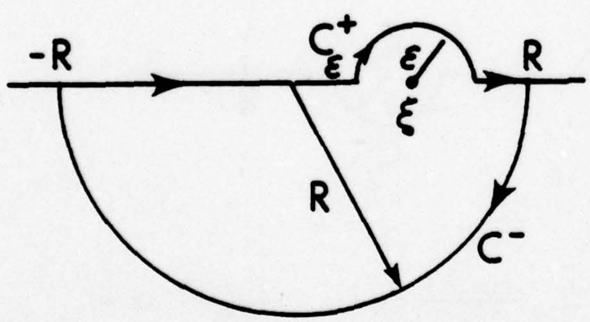

Fig. 9

Here requirement (ii-a) is changed into: (ii-b) $|f(z,\lambda)| \to 0$ as $|z| \to \infty$, uniformly in arg z, for $\pi \leq \arg z = \theta \leq 2\pi$. For $f(x,\lambda) = e^{-i\lambda x}\, g(x)$, with $\lambda > 0$ and $g(z)$ analytic, condition (ii-b) on f translates into: $|g(z)| \to$ constant as $|z| \to \infty$, uniformly in arg z, for $\pi \leq \arg z \leq 2\pi$. Again, this is a modified form of Jordan's lemma.

Since the direction of integration here is opposite to that of the previous case, being in fact the opposite to the conventional positive orientation of the contour used in Cauchy's residue theorem, we obtain

$$\int_{-R}^{\xi-\epsilon} \frac{f(x,\lambda)}{x-\xi}\,dx + \int_{\xi+\epsilon}^{R} \frac{f(x,\lambda)}{x-\xi}\,dx + \int_{C_\epsilon^+} \frac{f(z,\lambda)}{z-\xi}\,dz +$$

$$+\int_{C^-} \frac{f(z,\lambda)}{z-\xi}\,dz = -2\pi i \left[f(\xi,\lambda) + \sum_j \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_j} \right] \tag{I-8}$$

Using (ii-b) and taking the limits $\epsilon \to 0$ and $R \to \infty$ yields the result for this case

$$P\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\,dx - i\pi f(\xi,\lambda) = -2\pi i \left[f(\xi,\lambda) + \sum_j \operatorname{Res} \frac{f(z,\lambda)}{z-\xi} \Bigg|_{z=z_j} \right] \tag{I-9}$$

An identical result may be arrived at, if instead of the upper semicircle C_ϵ^+, we choose the lower semicircle C_ϵ^-. In this case the closed contour does not include the pole at $z = \xi$, but then C_ϵ^- gives, for $\epsilon \to 0$, the correct contribution.

It is also important to remark that the two PV's appearing in eqs. (I-5) and (I-9), should not be identified, since those two belong to different functions. One has the requisite properties in the upper half-plane while the other possesses them in the lower half of the z-plane.

We notice that the l.h.s. of eqs. (I-5) and (I-9) represent precisely Plemelj's integrals I_u and I_ℓ shown in eqs. (13) and (14). Therefore we can summarize all of the above results as:

$$P\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\,dx = \begin{cases} i\pi f(\xi,\lambda) + 2\pi i \sum_k \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_k} \\ \qquad \text{for f satisfying ii-a)} \\ -i\pi f(\xi,\lambda) - 2\pi i \sum_j \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_j} \\ \qquad \text{for f satisfying ii-b)} \end{cases} \tag{I-10}$$

$$I_u \equiv \int \frac{f(z,\lambda)}{z-\xi}\,dz = \begin{cases} 2\pi i \sum_k \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_k} \\ \qquad \text{for f satisfying ii-a)} \\ -2\pi i \left[f(\xi,\lambda) + \sum_j \operatorname{Res} \left.\frac{f(z,\lambda)}{z-\xi}\right|_{z=z_j} \right] \\ \qquad \text{for f satisfying ii-b)} \end{cases} \tag{I-11}$$

$$I_\ell \equiv \int \frac{f(z,\lambda)}{z-\xi}\,dz = \begin{cases} 2\pi i \quad f(\xi,\lambda) + \sum\limits_k \mathrm{Res}\, \left.\dfrac{f(z,\lambda)}{z-\xi}\right|_{z=z_k} : \\ \qquad \text{for f satisfying ii-a)} \\ -\,2\pi i \sum\limits_j \mathrm{Res}\, \left.\dfrac{f(z,\lambda)}{z-\xi}\right|_{z=z_j} \\ \qquad \text{for f satisfying (ii-b)} \end{cases} \tag{I-12}$$

It is important to observe that Plemelj's integrals I_u and I_ℓ, which originally were defined along indented paths, with the indentations ultimately shrunk to zero, can be evaluated by contour integrations (whenever the appropriate conditions are satisfied) as if the indentations were fixed. That this is true can be immediately seen from the r.h.s. of eqs. (I-11) and (I-12). No reference is made there to the PV's of the integrals (although these PV's enter the result implicitly). Hence I_u and I_ℓ can be interpreted also as integrals along indented, but fixed, paths and their evaluation performed by contour integration. Such integrals, which avoid the singularities in their integrands, are familiar from the theory of the inverse Laplace transform (where, however, the paths separate the z-plane into left and right halves).

A further remark concerns the frequently met case when $f(x,\lambda)$ is a real function. It may happen then that the extension of $f(x,\lambda)$ to the complex plane, namely $f(z,\lambda)$, does not satisfy either one of the requirements (ii-a) or (ii-b). Under these circumstances it would appear that the PV cannot be evaluated by contour integration. Nevertheless, this difficulty may be circumvented if it is possible to find a complex function $F(z,\lambda)$ which satisfies at least one of these requirements, and such that $f(x,\lambda)$ is equal to the real or imaginary part of $F(z,\lambda)$ in the limit $y \to 0$. Two well-known examples are $f_1(x,\lambda) = \sin \lambda x$ and $f_2(x,\lambda) = J_n(\lambda x)$. The appropriate functions are: $F_1(z,\lambda) = \exp(i\lambda z)$ with $f_1(x,\lambda) = \mathrm{Im}\,\{F_1(x,\lambda)\}$, and $F_2(x,\lambda) = H_n^{(1)}(\lambda z)$ with $f_2(x,\lambda) = \mathrm{Re}\,\{F_2(x,\lambda)\}$. Use of F_1 and F_2 in the contour integration will ultimately yield the desired results for f_1 and f_2.

When contour integration is not convenient or impossible, we can go back to the original definition of the PV. Such cases may occur when $f(z,\lambda)$ has branch points in the z-plane or when it is not analytic. An example for the latter case is $f(z,\lambda) = \exp(-\lambda|z|)$, $\lambda > 0$.

A simple procedure is useful in obtaining the PV exactly or, at the very least, an alternative expression for it that does not involve the limit process $\varepsilon \to 0$. This is based on the fact that any function $g(x)$ may be represented as a sum of two functions, one odd and the other even:

$$g(x) = \frac{1}{2}\left[g(x) - g(-x)\right] + \frac{1}{2}\left[g(x) + g(-x)\right] \tag{I-13}$$

To use this, we change the variable of integration in the integral to obtain

$$\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx = \int_{-\infty}^{\infty} \frac{f(x+\xi,\lambda)}{x}\, dx \tag{I-14}$$

Identifying $g(x)$ with $f(x+\xi,\lambda)$ we find

$$\int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx = \int_{-\infty}^{\infty} \frac{f(x+\xi,\lambda) - f(-x+\xi,\lambda)}{2x}\, dx +$$

$$+ \int_{-\infty}^{\infty} \frac{f(x+\xi,\lambda) + f(-x+\xi,\lambda)}{2x}\, dx \tag{I-15}$$

The first integral on the r.h.s. of (I-15) has an even integrand. Hence, if this integral exists, it does not vanish. Moreover, the integrand tends to $f'(\xi)$ when $x \to 0$, and if $f'(\xi)$ is finite, there is no singularity at the origin. Therefore, with appropriate behavior of $f(x,\lambda)$ at $x = \pm \infty$, the first integral will exist in the conventional sense. On the other hand, the integrand in the second integral has a singularity at $x = 0$, but because the integrand is odd, the definition of the PV yields zero for this integral.

Finally, we may write

$$P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx = \int_{-\infty}^{\infty} \frac{f(x+\xi,\lambda) - f(-x+\xi,\lambda)}{2x}\, dx \tag{I-16}$$

Apart from a numerical factor, eq. (I-16) was obtained by Hilbert (Ch. V of Ref. 15).

Thus we see that the PV is given bv a conventional integral, which may still be difficult to evaluate, but at least the limit process $\varepsilon \to 0$ has been thereby eliminated. The same is true for Plemelj's formulas (13) and (14).

A method somewhat similar to the above is to subtract from (I-1) the singular integral $P \int_{-\infty}^{\infty} f(\xi,\lambda)/(x-\xi)(dx)$, whose value is obviously zero. Then we obtain on the r.h.s. a conventional integral

$$P \int_{-\infty}^{\infty} \frac{f(x,\lambda)}{x-\xi}\, dx = \int_{-\infty}^{\infty} \frac{f(x,\lambda) - f(\varepsilon,\lambda)}{x-\varepsilon}\, dx \tag{I-17}$$

provided $f'(\xi,\lambda)$ is finite for the values of ξ and λ considered.

APPENDIX II

LIST OF FULL-RANGE SINGULAR INTEGRALS

In this Appendix we present a list of the values that can be associated with the singular integrals of several functions. This list is constructed by showing, in some detail, how to apply the theory developed in the text to the various functions under consideration. Here the particular values used are all PV's and the additive term of the δ-function interpretation are understood, but not shown.

1. $\underline{f(x,\lambda) = e^{i\lambda x}}$

If $\lambda > 0$, $e^{i\lambda x}$ satisfies (ii-a) of Appendix I and we close the path in the upper half of the z-plane. Since $e^{i\lambda x}$ is analytic and has no poles in the entire finite plane, eq. (I-5) yields

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi e^{i\lambda\xi}\;; \quad \lambda > 0 \tag{II-1}$$

Similarly if $\lambda < 0$, $f(x,\lambda)$ satisfies (ii-b) and eq. (I-8) is the one applicable here. Then we get

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-\xi}\,dx = -\,i\pi e^{i\lambda\xi}\;; \quad \lambda < 0 \tag{II-2}$$

When $\lambda = 0$, $f(x,\lambda) = 1$, and although analytic, this function does not satisfy the criteria (ii-a) or (ii-b). Therefore, instead of contour integration, we go directly to the definition (I-1) to compute

$$P\int_{-\infty}^{\infty} \frac{dx}{x-\xi} = \lim_{R\to\infty}\;\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \Big[(\ln|-\varepsilon| - \ln|-R|) + (\ln R - \ln\varepsilon)\Big] = 0 \tag{II-3}$$

Now we can combine (II-1), (II-2) and (II-3) to write

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi} \tag{II-4}$$

where

$$\mathrm{sgn}(u) = \begin{cases} +1 & u > 0 \\ 0 & u = 0 \\ -1 & u < 0 \end{cases} \tag{II-5}$$

Eq. (II-4) is valid for <u>all</u> real λ and ξ. Separating the real and imaginary parts of (II-4), we obtain two other results:

$$P\int_{-\infty}^{\infty} \frac{\cos\lambda x}{x-\xi}\, dx = -\pi \,\mathrm{sgn}(\lambda)\sin(\lambda\xi)$$

$$= -\pi \sin(|\lambda|\xi) \tag{II-6}$$

$$P\int_{-\infty}^{\infty} \frac{\sin\lambda x}{x-\xi}\, dx = \pi \,\mathrm{sgn}(\lambda)\cos(\lambda\xi) \tag{II-7}$$

The last two results are again valid for all real λ and ξ. If in (II-6) we let $\xi \to 0$ we obtain

$$P\int_{-\infty}^{\infty} \frac{\cos\lambda x}{x}\, dx = 0 \tag{II-8}$$

This result can be also obtained directly from the definition of a PV. Similarly, if we let $\xi \to 0$ in (II-7) we obtain

$$\int_{-\infty}^{\infty} \frac{\sin\lambda x}{x}\, dx = \pi\,\mathrm{sgn}(\lambda) \tag{II-9}$$

Here the PV designation has been deleted since the integral in (II-9) is known to converge (in fact the integrand is not singular at $x = 0$) and has been evaluated many times by purely real methods (Ref. 8). This is not the case for the integrand in (II-8), although the integrals

$$\int_{a}^{\infty} \frac{\cos\lambda x}{x}\, dx, \quad \int_{-\infty}^{-b} \frac{\cos\lambda x}{x}\, dx$$

are known to converge for a, b > 0.

It should be noted that making use of (II-8) and (II-9), we could obtain through a change in the variable of integration, the results in (II-6), (II-7) and ultimately (II-4). While this confirms the discussion preceding eq. (I-16), it should be obvious that the present procedure is much faster, even for the simple case considered above.

Plemelj's formulas (13) and (14) yield the following results:

$$\int \frac{e^{i\lambda z}}{z-\xi}\, dz = \begin{cases} 0 & \lambda \geq 0 \\ -2\pi i e^{i\lambda\xi} & \lambda < 0 \end{cases}$$

$$= -2\pi i e^{i\lambda\xi} H(-\lambda) \tag{II-10}$$

where

$$H(u) = \begin{cases} 0 & u < 0 \\ 1 & u > 0 \end{cases} \tag{II-11}$$

Similarly

$$\int_{\smile\!\rightarrow} \frac{e^{i\lambda z}}{z-\xi}\, dz = 2\pi i e^{i\lambda\xi}\, H(\lambda) \tag{II-12}$$

Separation of real and imaginary parts in (II-10) and (II-12) yields

$$\int_{\frown\!\rightarrow} \frac{\cos\lambda z}{z-\xi}\, dz = 2\pi \sin(\lambda\xi) H(-\lambda) \tag{II-13}$$

$$\int_{\smile\!\rightarrow} \frac{\cos\lambda z}{z-\xi}\, dz = -2\pi \sin(\lambda\xi) H(\lambda) \tag{II-14}$$

$$\int_{\frown\!\rightarrow} \frac{\sin\lambda z}{z-\xi}\, dz = -2\pi \cos(\lambda\xi) H(-\lambda) \tag{II-15}$$

$$\int_{\smile\!\rightarrow} \frac{\sin\lambda z}{z-\xi}\, dz = 2\pi \cos(\lambda\xi) H(\lambda) \tag{II-16}$$

A host of related results may be obtained from (II-4), (II-10) and (II-12), if we admit now generalized functions as integrands. The integrals under consideration are highly divergent, even when the singularity at $x = \xi$ is absent and the conventional PV does not exist. But we can extend the PV definition to cover generalized functions as well, such that the PV's of these integrals will themselves be generalized functions of the appropriate parameters.

To illustrate these ideas let us start with eq. (II-6) and differentiate it once with respect to λ. By the preceding discussion, the reversal of the P-operation and differentiation in the l.h.s. is allowed and we find:

$$\begin{aligned} P \int_{-\infty}^{\infty} \frac{x e^{i\lambda x}}{x-\xi}\, dx &= \pi \frac{d}{d\lambda} \left[\mathrm{sgn}(\lambda) e^{i\lambda\xi} \right] \\ &= \pi \left[2\delta(\lambda) + i\xi\, \mathrm{sgn}(\lambda) \right] e^{i\lambda\xi} \end{aligned} \tag{II-17}$$

where $d/d\lambda(\mathrm{sgn}(\lambda)) = 2\delta(\lambda)$.

The validity of interchanging differentiation and the P-operation can be verified by evaluating directly the l.h.s. of (II-17)

$$P\int_{-\infty}^{\infty} \frac{xe^{i\lambda x}}{x-\xi}\,dx = P\int_{-\infty}^{\infty} e^{i\lambda x}\,dx + \xi P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-\xi}\,dx$$

and this is seen to be identical with the r.h.s. of (II-17).

Proceeding in this fashion we can write

$$P\int_{-\infty}^{\infty} \frac{x^k e^{i\lambda x}}{x-\xi}\,dx = \left(\frac{1}{i}\frac{d}{d\lambda}\right)^k \left\{\pi i\,\mathrm{sgn}(\lambda)e^{i\lambda\xi}\right\} \qquad \text{(II-18)}$$

Moreover, if $f_N(x)$ is an N-th order polynomial we obtain the result

$$P\int_{-\infty}^{\infty} \frac{f_N(x)e^{i\lambda x}}{x-\xi}\,dx = f_N\left(\frac{1}{i}\frac{d}{d}\right)\left\{\pi i\,\mathrm{sgn}(\lambda)e^{i\lambda\xi}\right\} \qquad \text{(II-19)}$$

If the nonpolynomial ordinary function f(x) is such that it exists as a generalized function (Ref. 6), then (II-19) can be extended as follows

$$P\int_{-\infty}^{\infty} \frac{f(x)e^{i\lambda x}}{x-\xi}\,dx = f\left(\frac{1}{i}\frac{d}{d\lambda}\right)\left\{\pi i\,\mathrm{sgn}(\lambda)e^{i\lambda\xi}\right\}$$

An example of a function that does not satisfy the above is $f(x) = e^{-\mu x}$, $\mu > 0$, $|x| \leq \infty$ since for $x < 0$, this is not a generalized function.

Similar expressions can be set down for (II-10) and (II-12) but we shall not do so here.

2. $f(x,\lambda) = e^{i\lambda x}$ and two poles $\xi_1 = a$; $\xi_2 = b$

$$P\int_{-\infty}^{\infty} \frac{e^{\lambda i x}}{(x-a)(x-b)} = \frac{1}{a-b}\left[P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-a}\,dx - P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-b}\,dx\right]$$

$$= \pi i\,\mathrm{sgn}(\lambda)\,\frac{e^{i\lambda a} - e^{i\lambda b}}{a-b}\ ;\ a \neq b \qquad \text{(II-20)}$$

The real and imaginary parts of (II-20) yield

$$P\int_{-\infty}^{\infty} \frac{\cos\lambda x}{(x-a)(x-b)}\,dx = -\pi\,\mathrm{sgn}(\lambda)\,\frac{\sin\lambda a - \sin\lambda b}{a-b} \qquad \text{(II-21}_o\text{)}$$

$$P\int_{-\infty}^{\infty} \frac{\sin\lambda x}{(x-a)(x-b)}\,dx = \pi\,\mathrm{sgn}(\lambda)\,\frac{\cos\lambda a - \cos\lambda b}{a-b} \qquad \text{(II-21}_1\text{)}$$

We assume in the above that at least one of the two quantities a,b is nonzero.

The particular case $b = -a \neq 0$ arises in many applications and therefore we set it down

$$P \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2-a^2}\, dx = -\pi \operatorname{sgn}(\lambda)\, \frac{\sin\lambda a}{a} \tag{II-22}$$

Hence

$$P \int_{-\infty}^{\infty} \frac{\cos\lambda x}{x^2-a^2}\, dx = -\pi \operatorname{sgn}(\lambda)\, \frac{\sin\lambda a}{a}$$

$$= -\pi\, \frac{\sin(|\lambda| a)}{a} \tag{II-23}$$

The case $\lambda = 0$ yields, from eqs. (II-22), (II-23), the zero value for the PV. But if we admit the possibility that the arbitrary parameter a may take the value a = 0, we may ask what should be the interpretation of the integral $\int_{-\infty}^{\infty} dx/(x^2-a^2)$. We address this question in Appendix VII since it affords a good example of the δ-function interpretation and also because this integral has been discussed previously in the literature.

We have also

$$\int \frac{e^{i\lambda x}}{(x-a)(x-b)}\, dx = -2\pi i H(-\lambda)\, \frac{e^{i\lambda a} - e^{i\lambda b}}{a-b} \tag{II-24}$$

$$\int \frac{e^{i\lambda x}}{x^2-a^2}\, dx = 2\pi H(-\lambda)\, \frac{\sin\lambda a}{a} \tag{II-25}$$

$$\int \frac{e^{i\lambda x}}{(x-a)(x-b)}\, dx = 2\pi i H(\lambda)\, \frac{e^{i\lambda a} - e^{i\lambda b}}{a-b} \tag{II-26}$$

$$\int \frac{e^{i\lambda x}}{x^2-a^2}\, dx = -2\pi H(\lambda)\, \frac{\sin\lambda a}{a} \tag{II-27}$$

Since we have two poles, the Plemelj paths may be chosen in four different ways, two as in (II-24) and (II-26) and the other two as follows

$$\int \frac{e^{i\lambda x}}{(x-a)(x-b)}\,dx = -2\pi i\,\frac{e^{i\lambda a}H(-\lambda) + e^{i\lambda b}H(\lambda)}{a-b} \qquad \text{(II-28)}$$

$$\int \frac{e^{i\lambda x}}{(x-a)(x-b)}\,dx = 2\pi i\,\frac{e^{i\lambda a}H(\lambda) + e^{i\lambda b}H(-\lambda)}{a-b} \qquad \text{(II-29)}$$

and

$$\int \frac{e^{i\lambda x}}{x^2-a^2}\,dx = -\pi i\,\frac{e^{i\lambda a}H(-\lambda) + e^{-i\lambda a}H(\lambda)}{a} \qquad \text{(II-30)}$$

$$\int \frac{e^{i\lambda x}}{x^2-a^2}\,dx = \pi i\,\frac{e^{i\lambda a}H(\lambda) + e^{-i\lambda a}H(-\lambda)}{a} \qquad \text{(II-31)}$$

It is of interest to note that eqs. (II-25) and (II-27) have no imaginary parts and therefore we can write

$$\int \frac{\cos\lambda x}{x^2-a^2}\,dx = 2\pi H(-\lambda)\,\frac{\sin\lambda a}{a} \qquad \text{(II-32)}$$

$$\int \frac{\cos\lambda x}{x^2-a^2}\,dx = -2\pi H(\lambda)\,\frac{\sin\lambda a}{a} \qquad \text{(II-33)}$$

3. $f(x,\lambda) = e^{i\lambda x}$ and M Poles

Here we want to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g_M(x)}\,dx \qquad \text{(II-34)}$$

where $g_M(x)$ is an M-th order polynomial with real coefficients, such that all of its M roots are simple and real. If these roots are denoted by x_m a well-known formula of partial fraction decomposition (PFD) yields:

$$\frac{1}{g_M(x)} = \sum_{m=1}^{M} \frac{1}{g_M'(x_m)}\,\frac{1}{x-x_m}$$

Then we immediately find

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g_M(x)}\,dx = \pi\, i\,\mathrm{sgn}(\lambda) \sum_{m=1}^{M} \frac{1}{g_M'(x_m)}\, e^{i\lambda x_m} \tag{II-35}$$

The analogue of eq. (II-19) is here

$$P\int_{-\infty}^{\infty} \frac{f_N(x)e^{i\lambda x}}{g_M(x)}\,dx = f_N\left(\frac{1}{i}\frac{d}{d\lambda}\right)\left\{\pi\, i\,\mathrm{sgn}(\lambda) \sum_{m=1}^{M} \frac{e^{i\lambda x_m}}{g_M'(x_m)}\right\} \tag{II-36}$$

If the roots of the polynomial $g_M(x)$ are not all real, the PV can still be found, using the PFD, provided the locations of the complex roots of $g_M(x)$ are known. In this case contour integration yields rapidly the desired result.

Since there are M poles in (II-34), the number of possible Plemelj paths is 2^M, and hence there are 2^M possible Plemelj values for this integral. It is clear that for each of these Plemelj paths we can write down formulas analogous to (II-36).

4. $f(x,\lambda) = e^{i\lambda x}$ and an infinity of poles

The integral to be considered here is

$$I = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g(x)}\,dx \tag{II-37}$$

where g(x) is taken to be a real function with an infinite number of roots, all of which are real and simple.

First we remark that for $\lambda = 0$, the PV of (II-37) will be zero if g(x) is an odd function. In the opposite case of even g(x), the PV may or may not exist, depending on the detailed behavior of g(x). It remains to discuss the cases $\lambda > 0$ and $\lambda < 0$.

If we denote the roots of g(x) by x_m we can evaluate the PV by contour integration as in the case of a single pole. Since by assumption g(x) has no complex roots we obtain

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{e^{i\lambda x}}{g(x)}\,dx + \lim_{\epsilon\to 0^+} \sum_m i\int_{\pi}^{0} \frac{\epsilon e^{i\psi} e^{i\lambda(x_m+\epsilon e^{i\psi})}}{g(x_m+\epsilon e^{i\psi})}\,d\psi$$

$$+ \lim_{R\to\infty} i\int_0^{\pi} \frac{Re^{i\theta}\; e^{i\lambda Re^{i\theta}}}{g(Re^{i\theta})}\,d\theta = 0 \tag{II-38}$$

This is the result of closing the path of integration in the upper half plane, for $\lambda > 0$.

In order that the third integral vanish in the limit $R \to \infty$, it will be sufficient to have $1/|g(z)| \to$ a constant, as $|z| \to \infty$, for all arg z, such that $0 \leq \arg z \leq \pi$. In particular the above constant may be zero. This is due to the fact that in the numerator we have $\exp(-\lambda R\sin\theta) \to 0$ for $R \to \infty$. Hence we obtain from (II-38), using L'Hospital's theorem,

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g(x)}\, dx = \pi i \sum_m \frac{e^{i\lambda x_m}}{g'(x_m)} \;;\quad \lambda > 0 \tag{II-39}$$

If we assume that $|g(z)|^{-1} \to$ a constant as $|z| \to \infty$, uniformly in arg z, for $\pi \leq \arg z \leq 2\pi$, we can extend the result to the lower half plane, to obtain

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g(x)}\, dx = -\pi i \sum_m \frac{e^{i\lambda x_m}}{g'(x_m)} \;;\quad \lambda < 0 \tag{II-40}$$

For an odd function g(x), we can combine these results into

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{g(x)}\, dx = \pi i\,\mathrm{sgn}(\lambda) \sum_m \frac{e^{i\lambda x_m}}{g'(x_m)} \tag{II-41}$$

If the infinite sum diverges, we may still interpret it as a generalized function of λ and as such it will be useful when appearing as the kernel of an integral operator.

This result suggests that we may extend formally the formula for the PFD of a polynomial to an arbitrary function, even or odd. To see this, we write for arbitrary g(x)

$$\frac{1}{g(x)} = \sum_m \frac{1}{g'(x_m)} \frac{1}{x-x_m} \tag{II-42}$$

where the roots x_m may be arbitrary complex numbers.

The validity of (II-42) and its extensions are discussed in Ref. 10. When we substitute (II-42) into (II-37) and use (II-4) we obtain precisely (II-41). Notice that nothing was said about the convergence of the sum in (II-42).

We can immediately apply (II-41) to several functions whose zeros are known explicitly.

$$P \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\sin x} dx = \pi i \operatorname{sgn}(\lambda) \sum_{m=-\infty}^{\infty} (-1)^m e^{i\lambda m\pi} \tag{II-43}$$

The function $g(x) = \sin x$ has as zeros $x_m = m\pi$; $m = 0, \pm 1, \pm 2, \ldots$ and $g'(x_m) = (-1)^m$. As a matter of fact the PFD of $(\sin x)^{-1}$ is well-known (§ 93 of Ref. 9), and is written in convergent form as

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{2x}{x^2 - n^2\pi^2}$$

which follows directly also from (II-42).

The sum in (II-43) is a generalized function which can be written in a more familiar form, when we use certain results of Refs. (4) and (6),

$$P \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\sin x} dx = 2\pi i \operatorname{sgn}(\lambda) \sum_{k=-\infty}^{\infty} \delta(\lambda-(2k+1)) \tag{II-44}$$

A further verification of (II-44) may be obtained from the Fourier series expansion of the generalized function $(\sin x)^{-1}$. It can be shown that this (generalized) expansion is given by

$$\frac{1}{\sin x} = 2 \sum_{k=0}^{\infty} \cdot \sin(2k+1)x \tag{II-45}$$

When we substitute (II-45) into (II-37) and perform the integration term by term, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\sin x} dx = 2\pi i \sum_{k=0}^{\infty} \left[\delta(\lambda-2k-1) - \delta(\lambda+2k+1)\right] \tag{II-46}$$

But the r.h.s. of (II-46) is identical with the r.h.s. of (II-44). This result seems to imply that the Fourier transform of the generalized function $(\sin x)^{-1}$ is identical with the PV of the Fourier transform of the ordinary function $(\sin x)^{-1}$. Since the following examples exhibit the same property, it may be conjectured that this is true of all ordinary functions whose definition can be extended nontrivially to become generalized functions.

The function $g(x) = \cos x$ has the zeros $x_m = (m+1/2)\pi$, $m = 0, \pm 1, \pm 2, \ldots$, and $g'(x_m) = (-1)^{m+1}$.

Therefore

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\cos x}\,dx = -\pi i\,\mathrm{sgn}(\lambda) \sum_{m=-\infty}^{\infty} (-1)^m\, e^{i\lambda(m+\frac{1}{2})\pi}$$

$$= 2\pi\,\mathrm{sgn}(\lambda) \sum_{k=-\infty}^{\infty} (-1)^k\, \delta(\lambda-(2k+1)) \qquad \text{(II-47)}$$

Now, the Fourier series expansion of the generalized function $(\cos x)^{-1}$ is

$$\frac{1}{\cos x} = 2 \sum_{k=0}^{\infty} (-1)^k \cos(2k+1)x \qquad \text{(II-48)}$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\cos x}\,dx = 2\pi \sum_{k=0}^{\infty} (-1)^k \left[\delta(\lambda+2k+1) + \delta(\lambda-2k-1)\right] \qquad \text{(II-49)}$$

It is easily seen that the r.h.s. of (II-47) is identical with the r.h.s. of (II-49), in accord with the conjecture stated above.

The zeros of the function $g(x) = \tan x$ are $x_m = m\pi$, $m = 0, \pm 1, \pm 2, \ldots$, and $g'(x_m) = 1$. Then we find

$$P\int_{-\infty}^{\infty} e^{i\lambda x} \cot x\,dx = i\pi\,\mathrm{sgn}(\lambda) \sum_{m=-\infty}^{\infty} e^{i\lambda m\pi}$$

$$= 2\pi i\,\mathrm{sgn}(\lambda) \sum_{k=-\infty}^{\infty} \delta(\lambda-2k) \qquad \text{(II-50)}$$

Since the Fourier series expansion of the generalized function cotx is

$$\cot x = 2 \sum_{k=1}^{\infty} \sin 2kx \qquad \text{(II-51)}$$

we obtain

$$\int_{-\infty}^{\infty} e^{i\lambda x} \cot x \, dx = 2\pi i \sum_{k=1}^{\infty} \left[\delta(\lambda-2k) - \delta(\lambda+2k) \right] \tag{II-52}$$

which again is seen to be identical with (II-50).

The zeros of the function $g(x) = \cot x$ are $x_m = (m + 1/2)\pi$, $m = 0, \pm 1, \pm 2, \ldots$, and $g'(x_m) = -1$. Then

$$P \int_{-\infty}^{\infty} e^{i\lambda x} \tan x \, dx = -\pi i \operatorname{sgn}(\lambda) \sum_{m=-\infty}^{\infty} e^{i\lambda(m+\frac{1}{2})\pi}$$

$$= -2\pi i \operatorname{sgn}(\lambda) \sum_{k=-\infty}^{\infty} (-1)^k \delta(\lambda-2k) \tag{II-53}$$

The Fourier series expansion of the generalized function tanx is

$$\tan x = -2 \sum_{k=1}^{\infty} (-1)^k \sin 2kx \tag{II-54}$$

and we find

$$\int_{-\infty}^{\infty} e^{i\lambda x} \tan x \, dx = -2\pi i \sum_{k=1}^{\infty} \left[\delta(\lambda-2k) - \delta(\lambda+2k) \right] \tag{II-55}$$

Once more, (II-55) is identical with (II-53).

All of the results (II-43) - (II-55) may be generalized, by analogy with (II-19) and the equation following it, when $e^{i\lambda x}$ is multiplied by a generalized function $f(x)$, in the respective integrals.

It is now easy to write the corresponding expressions for the Plemelj integrals if we make use of the above results. It is clear that there is an infinity of values that can be associated with each integral, since each of these values corresponds to a particular Plemelj path.

We give one example where the function $g(x)$ has an infinity of zeros, but only one of these is real, namely $g(x) = e^x - 1$. Here the zeros are $x_o = 0$ (the real root) and $x_m = 2\pi i m$, $m = \pm 1, \pm 2, \ldots$. Also we have $g'(x_o) = 1$; $g'(x_m) = 1$. Then the PFD of $(\bar{g}(x))^{-1}$ is

$$\frac{1}{e^x - 1} = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{m=1}^{\infty} \frac{x}{x^2 + (2\pi m)^2} \tag{II-56}$$

Therefore we obtain

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{e^{x}-1}\,dx = -P\int_{-\infty}^{\infty} \frac{1}{2}\, e^{i\lambda x}\,dx + P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x}\,dx + 2\sum_{m=1}^{\infty} P\int_{-\infty}^{\infty} \frac{xe^{i\lambda x}}{x^{2}+(2\pi m)^{2}}\,dx \tag{II-57}$$

We can show by contour integration and differentiation with respect to λ that, for $p > 0$

$$P\int_{-\infty}^{\infty} \frac{xe^{i\lambda x}}{x^{2}+p^{2}}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{-|\lambda|p}; \quad p > 0 \tag{II-58}$$

where P refers to the infinite limits of integration. Using the above we find

$$P\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{e^{x}-1}\,dx = -\pi\delta(\lambda) + i\pi\,\mathrm{sgn}(\lambda) + 2\pi i\,\mathrm{sgn}(\lambda) \sum_{m=1}^{\infty} e^{-2\pi|\lambda|m}$$

$$= -\pi\delta(\lambda) + i\pi\,\mathrm{sgn}(\lambda)\left[1 + 2\,\frac{1}{1-e^{-2\pi|\lambda|}} - 2\right]$$

$$= -\pi\delta(\lambda) + i\pi\,\mathrm{sgn}(\lambda)\coth(\pi|\lambda|) \tag{II-59}$$

Eq. (II-59) implies also that for the integral of the generalized function appearing below we have

$$P\int_{-\infty}^{\infty} \frac{\sin\lambda x}{e^{x}-1}\,dx = \pi\,\mathrm{sgn}(\lambda)\coth(\pi|\lambda|) \tag{II-60}$$

which may be written also as

$$\int_{0}^{\infty} \sin\lambda x \coth\frac{x}{2}\,dx = \pi\,\mathrm{sgn}(\lambda)\coth(\pi|\lambda|) \tag{II-61}$$

It is of interest to note that the integral of the same function on the half range is

$$\int_{0}^{\infty} \frac{\sin\lambda x}{e^{x}-1}\,dx = \frac{\pi}{2}\left[\coth(\pi\lambda) - \frac{1}{\pi\lambda}\right] \tag{II-62}$$

obtained here through use of the PFD (II-56) and of the result known from generalized function theory that

$$\int_0^\infty \sin\lambda x\, dx = \begin{cases} \frac{1}{\lambda}\ ; & \lambda \neq 0 \\ 0\ ; & \lambda = 0 \end{cases} \qquad \text{(II-63)}$$

The result (II-62) can be obtained also by conventional means since the integrand is an ordinary function without any singularities (see Ref. 9, p. 468). This again confirms the legitimacy of the PFD and its use under the integral sign.

Furthermore, we find from (II-59)

$$P \int_{-\infty}^{\infty} \frac{\cos\lambda x}{e^x - 1}\, dx = -\pi\delta(\lambda) \qquad \text{(II-64)}$$

It is not difficult to generalize the results of (II-56) - (II-64), when $g(x)$ is replaced by $e^x - a$, $0 < a$, but this is left to the reader.

5. $f(x,\lambda) = e^{-\lambda x^2};\ \lambda > 0$

The PV of the corresponding integral cannot be evaluated by contour integration. Hence we use (I-16) to obtain

$$P \int_{-\infty}^{\infty} \frac{e^{-\lambda x^2}}{x-\xi}\, dx = -e^{-\lambda\xi^2} \int_{-\infty}^{\infty} e^{-\lambda x^2} \frac{\sinh(2\lambda\xi x)}{x}\, dx \qquad \text{(II-65)}$$

The r.h.s. of (II-65) is obviously an ordinary integral, the even integrand has no singularities, and the convergence is very rapid.

6. $f(x) = e^{i(\mu x^2 + \lambda x)}$

Again, the PV of the following integral cannot be evaluated by contour integration and we have to resort to other means, such as (I-16), to evaluate it. But first we rewrite it as

$$P \int_{-\infty}^{\infty} \frac{e^{i(\mu x^2 + \lambda x)}}{x-\xi}\, dx = e^{-i\frac{\lambda^2}{4\mu}}\, P \int_{-\infty}^{\infty} \frac{e^{i\mu x^2}}{x-\zeta}\, dx \qquad \text{(II-66)}$$

where $\zeta = \xi + \lambda/2\mu$, and we have assumed that λ and $\mu \neq 0$, are arbitrary.

Use of (I-16) in the r.h.s. of (II-66) yields

$$P \int_{-\infty}^{\infty} \frac{e^{i\mu x^2}}{x-\zeta}\, dx = ie^{i\mu\zeta^2} \int_{-\infty}^{\infty} e^{i\mu x^2} \frac{\sin(2\mu\zeta x)}{x}\, dx \qquad \text{(II-67)}$$

The r.h.s. of (II-67) is a conventional integral with even integrand that has no singularities. Since a closed form analytic expression for it is not available, in any application concrete results will have to be obtained either numerically or by analytic approximation.

The real and imaginary parts of (II-67) yield

$$P \int_{-\infty}^{\infty} \frac{\cos(\mu x^2)}{x-\zeta}\, dx = -\sin(\mu\zeta^2) \int_{-\infty}^{\infty} \frac{\cos(\mu x^2)\sin(2\mu\zeta x)}{x}\, dx$$

$$- \cos(\mu\zeta^2) \int_{-\infty}^{\infty} \frac{\sin(\mu x^2)\sin(2\mu\zeta x)}{x}\, dx \qquad \text{(II-68)}$$

$$P \int_{-\infty}^{\infty} \frac{\sin(\mu x^2)}{x-\zeta}\, dx = \cos(\mu\zeta^2) \int_{-\infty}^{\infty} \frac{\cos(\mu x^2)\sin(2\mu\zeta x)}{x}\, dx$$

$$- \sin(\mu\zeta^2) \int_{-\infty}^{\infty} \frac{\sin(\mu x^2)\sin(2\mu\zeta x)}{x}\, dx \qquad \text{(II-69)}$$

APPENDIX III

INTEGRALS WITH MULTIPLE POLES

A list of such integrals can be easily constructed from the previous results by differentiation with respect to x the function $f(x,\lambda)$, and taking the PV of the resulting integral, as explained in Section III. Here we shall give only one example:

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x-\xi)^k}\, dx = \frac{1}{(k-1)!} (i\lambda)^{k-1}\; P \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x-\xi}\, dx$$

$$= \pi i^k \operatorname{sgn}(\lambda) \lambda^{k-1} e^{i\lambda\xi} \qquad \text{(III-1)}$$

Here $k > 0$ is an integer.

APPENDIX IV

HALF-RANGE INTEGRALS

In this Appendix we present several examples of PV's of half-range integrals. In all that follows we assume $\xi > 0$.

1. $f(x,\lambda) = e^{i\lambda x};\ \lambda \neq 0$

Use of eq. (73) and (II-4) for $F(\xi,\lambda)$, yields the expression

$$P\int_0^\infty \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi} + \int_0^\infty \frac{e^{-i\lambda x}}{x+\xi}\,dx$$

$$= i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi} + e^{i\lambda\xi}\int_\xi^\infty \frac{e^{-i\lambda x}}{x}\,dx \qquad \text{(IV-1)}$$

Taking real and imaginary parts we obtain

$$P\int_0^\infty \frac{\cos\lambda x}{x-\xi}\,dx = -\pi\sin(|\lambda|\xi) - \mathrm{ci}(|\lambda|\xi)\cos(\lambda\xi)$$

$$- \mathrm{si}(|\lambda|\xi)\sin(|\lambda|\xi) \qquad \text{(IV-2)}$$

$$P\int_0^\infty \frac{\sin\lambda x}{x-\xi}\,dx = \mathrm{sgn}(\lambda)\left[\pi\cos\lambda\xi + \cos(\lambda\xi)\mathrm{si}(|\lambda|\xi)\right.$$

$$\left.- \sin(|\lambda|\xi)\mathrm{ci}(|\lambda|\xi)\right] \qquad \text{(IV-3)}$$

where the functions ci(x) and si(x) are defined in § 8.23 of Ref. 13. Incidentally (IV-2) and (IV-3) are identical with the results given as No. 6 and No. 7 in § 3.722 of the same reference (the P symbol does not appear there, but it is clear that the results are PV integrals).

We remark that method (b) of contour integration is not applicable here. The reason is that the integral along the arc $(0, \alpha)$ increases without bounds when $|z| \to \infty$.

When eq. (80) of method (c) is used, we find, after a few easy calculations

$$P\int_0^\infty \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi} + \int_0^\infty \frac{e^{-u\xi}}{u+i\lambda}\,du \qquad \text{(IV-4)}$$

Since the PV of any integral, if it exists, is unique, the two integrals on the r.h.s. of (IV-1) and (IV-4) must be identical. This is easily proven, by using the Laplace tranform identity (76) for $(x+\xi)^{-1}$.

From (IV-4) we can also obtain

$$P\int_0^\infty \frac{\cos\lambda x}{x-\xi}\,dx = -\pi\,\mathrm{sgn}(\lambda)\sin\lambda\xi + \int_0^\infty \frac{ue^{-u\xi}}{u^2+\lambda^2}\,du \qquad \text{(IV-5)}$$

$$P\int_0^\infty \frac{\sin\lambda x}{x-\xi}\,dx = \pi\,\mathrm{sgn}(\lambda)\cos\lambda\xi - \lambda\int_0^\infty \frac{e^{-u\xi}}{u^2+\lambda^2}\,du \qquad \text{(IV-6)}$$

Comparing the results (IV-2), (IV-3) with (IV-5), (IV-6) respectively, we notice that the latter have monotonically decreasing integrands, and thus converge much faster than the former.

2. $\underline{f(x,\lambda) = e^{-\lambda x};\ \lambda > 0}$

Using eq. (80) we can show that the PV in this case is infinite. Indeed, we obtain

$$P\int_0^\infty \frac{e^{-\lambda x}}{x-\xi}\,dx = e^{-\lambda\xi}\int_0^\infty \frac{du}{u+\lambda} + \int_0^\infty \frac{e^{-u\xi}-e^{-\lambda\xi}}{u-\lambda}\,du \qquad \text{(IV-7)}$$

It is immediately apparent that the first integral on the r.h.s. of (IV-7) is divergent. The same conclusion can be obtained from eq. (74), since the integral along the ray $z = e^{i\alpha}y$, $y > 0$ and $0 < \alpha < \pi/2$, diverges.

3. $\underline{f(x,\lambda) = e^{-\lambda x^2};\ \lambda > 0}$

Use of method (a) yields here the result

$$P\int_0^\infty \frac{e^{-\lambda x^2}}{x-\xi}\,dx = -2e^{-\lambda\xi^2}\int_0^\infty e^{-\lambda x^2}\,\frac{\sinh(2\lambda\xi x)}{x}\,dx$$

$$+\int_0^\infty \frac{e^{-\lambda x^2}}{x+\xi}\,dx \qquad \text{(IV-8)}$$

We have used here (II-65).

An equivalent result may be also obtained by method (c), but (IV-8) seems to be the more rapidly convergent expression and therefore we omit it.

4. $f(x,\lambda) = e^{i\lambda x^2}$

Making use of (II-67) we can write

$$P\int_0^\infty \frac{e^{i\lambda x^2}}{x-\xi}\,dx = 2ie^{i\lambda\xi^2}\int_0^\infty e^{i\lambda x^2}\,\frac{\sin(2\lambda\xi x)}{x}\,dx$$

$$+\int_0^\infty \frac{e^{i\lambda x^2}}{x+\xi}\,dx \qquad \text{(IV-9)}$$

Taking real and imaginary parts we deduce

$$P\int_0^\infty \frac{\cos\lambda x^2}{x-\xi}\,dx = \int_0^\infty \frac{\cos\lambda x^2}{x+\xi}\,dx - 2\cos\lambda\xi^2\int_0^\infty \frac{\sin(\lambda x^2)\sin(2\lambda\xi x)}{x}\,dx$$

$$-2\sin\lambda\xi^2\int_0^\infty \frac{\cos(\lambda x^2)\sin(2\lambda\xi x)}{x}\,dx \qquad \text{(IV-10)}$$

$$P\int_0^\infty \frac{\sin\lambda x^2}{x-\xi}\,dx = \int_0^\infty \frac{\sin\lambda x^2}{x+\xi}\,dx + 2\cos\lambda\xi^2\int_0^\infty \frac{\cos(\lambda x^2)\sin(2\lambda\xi x)}{x}\,dx$$

$$-2\sin\lambda\xi^2\int_0^\infty \frac{\sin(\lambda x^2)\sin(2\lambda\xi x)}{x}\,dx \qquad \text{(IV-11)}$$

APPENDIX V

FINITE INTEGRALS

In this Appendix we present several illustrations of the results obtained in Section V. Here we assume $a < \xi < b$.

1. $f(x,\lambda) = e^{i\lambda x}$

Method (a), eq. (87), yields

$$P\int_a^b \frac{e^{i\lambda x}}{x-\xi}\,dx = \int_a^b \frac{e^{i\lambda x}-e^{i\lambda\xi}}{x-\xi}\,dx + e^{i\lambda\xi}\log\frac{b-\xi}{\xi-a}$$

$$= e^{i\lambda\xi}\left[2i\int_{\frac{a-\xi}{2}}^{\frac{b-\xi}{2}} e^{i\lambda t}\,\frac{\sin\lambda t}{t}\,dt + \log\frac{b-\xi}{\xi-a}\right] \qquad \text{(V-1)}$$

Contour integration yields, from eq. (88),

$$P\int_a^b \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\, e^{i\lambda\xi} + i\int_0^\pi \frac{e^{i\lambda\frac{b+a}{2}}\, e^{i\lambda\frac{b-a}{2}\, e^{i\theta}}}{1+\frac{b+a-2\xi}{b-a}\, e^{-i\theta}}\,d\theta \qquad \text{(V-2)}$$

Method (c), through eq. 89, implies that

$$P\int_a^b \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi}$$

$$-\int_0^\infty \frac{(u+i\lambda)e^{i\lambda b}e^{-(b-\xi)u} + (u-i\lambda)e^{i\lambda a}e^{-(\xi-a)u}}{u^2+\lambda^2}\,du \qquad \text{(V-3)}$$

The integral on the r.h.s. of (V-3) is a rapidly convergent integral. Furthermore, we notice that the first term on the r.h.s. is precisely the PV of the full-range integral.

Finally, we use directly the PV of the full-range integral, to write from eq. (90)

$$P\int_a^b \frac{e^{i\lambda x}}{x-\xi}\,dx = i\pi\,\mathrm{sgn}(\lambda)e^{i\lambda\xi} - \int_{-\infty}^a \frac{e^{i\lambda x}}{x-\xi}\,dx - \int_b^\infty \frac{e^{i\lambda x}}{x-\xi} \qquad \text{(V-4)}$$

If in (V-4) we use identity (76), we can show that (V-3) and (V-4) are indeed identical. But since the PV is unique (V-1) and (V-2) must have the same value. This implies that we can establish certain identities connecting the various integrals in the above four formulas. We notice that these identities are far from being trivial. By taking real and imaginary parts a number of other identities can be similarly obtained.

2. $f(x,\lambda) = e^{\lambda x}$; λ real

For the present example, methods (c) and (d) are not applicable, regardless of the values of λ. Therefore, we find from (a) and (b) the respective results:

$$P \int_a^b \frac{e^{\lambda x}}{x-\xi}\,dx = \int_a^b \frac{e^{\lambda x}-e^{\lambda\xi}}{x-\xi}\,dx + e^{\lambda\xi}\log\frac{b-\xi}{\xi-a}$$

$$= 2e^{\lambda\xi} \int_{\frac{a-\xi}{2}}^{\frac{b-\xi}{2}} e^{\lambda t}\,\frac{\sinh\lambda t}{t}\,dt \tag{V-5}$$

$$P \int_a^b \frac{e^{\lambda x}}{x-\xi}\,dx = i\pi e^{\lambda\xi} + i\int_0^{\pi} \frac{\exp\left(\lambda\frac{b+a}{2} + \lambda\frac{b-a}{2}e^{i\theta}\right)}{1 + \frac{b+a-2\xi}{b-a}e^{-i\theta}}\,d\theta \tag{V-6}$$

We note that the uniqueness of the PV again will yield interesting integral identities. One such identity is obtained immediately by equating to zero the imaginary part of the r.h.s. of (V-6), since the l.h.s. is real.

Other examples may be treated in a similar fashion, provided the methods used are applicable. That this is not always the case we have shown in example 2 above.

APPENDIX VI

EVALUATION OF $\int_0^\infty dx/(x-\xi)$

The integral

$$I(\xi) = \int_0^\infty \frac{dx}{x-\xi} \tag{VI-1}$$

where ξ is arbitrary, is a divergent integral by the conventional definition of Riemann integration as well as by its PV interpretation. Moreover, we note that the divergence is not due to the presence of the pole in the integrand, since the integral

$$I_a(\xi) = \int_a^\infty \frac{dx}{x-\xi} \tag{VI-2}$$

where $|\xi| < a$, is also divergent.

Therefore, the question arises whether one can attach a finite particular value to (VI-1). Now, we have shown in Section II(c) that $(x-\xi)^{-1}$ should be interpreted as

$$\left.\frac{1}{x-\xi}\right|_g = \left.\frac{1}{x-\xi}\right|_p + C\delta(x-\xi) \tag{32}$$

If we perform the integration indicated in (VI-1), we have

$$I^g(\xi) = I^p(\xi) + CH(\xi) \tag{VI-3}$$

To associate a finite value to $I^p(\xi)$, we borrow Hadamard's Finite Part idea (Ref. 11) to calculate it in the following fashion: we differentiate $I^p(\xi)$ formally with respect to ξ. The result is

$$\frac{d}{d\xi}(I^p(\xi)) = \int_0^\infty \frac{dx}{(x-\xi)^2} \tag{VI-4}$$

We perform the integration in the r.h.s. of (VI-4), since this is a convergent integral. Thus we obtain

$$\frac{d}{d\xi}(I^p(\xi)) = -\frac{1}{\xi} \tag{VI-5}$$

The final step is the integration of the differential equation (VI-5). Its solution is

$$I^{p}(\xi) = B - \log|\xi| \tag{VI-6}$$

where B is an arbitrary constant of integration.

Substitution of (VI-6) into (VI-3) yields the desired result

$$I^{g}(\xi) = B + CH(\xi) - \log|\xi| \tag{VI-7}$$

But we notice that, due to the properties of the Heaviside function $H(\xi)$, eq. (VI-7) can be written also as

$$I^{g}(\xi) = C_1 H(-\xi) + C_2 H(\xi) - \log|\xi| \tag{VI-8}$$

where C_1 and C_2 are arbitrary constants.

According to Hadamard, what we have done here was to discard the singularity present at upper endpoint of the interval of integration, namely at $x = \infty$.

It is of interest to note that we can obtain result (VI-6), apart from the arbitrary constant B, directly from the definition of PV, in conjunction with Hadamard's recipe of discarding the (infinite) contribution from the singularity at infinity.

Thus, we can write

$$P\int_0^\infty \frac{dx}{x-\xi} = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\left[\int_0^{\xi-\varepsilon}\frac{dx}{x-\xi} + \int_{\xi+\varepsilon}^{R}\frac{dx}{x-\xi}\right]$$

$$= \lim_{R\to\infty}\left[-\log|\xi| + \log|R-\xi|\right] \tag{VI-9}$$

Discarding the term $\log|R-\xi|$ which represents the infinite contribution (or replacing it with a finite constant B), we obtain the result in (VI-6). We notice one difference in the two procedures: in the first treatment ξ was arbitrary, which implied that for $\xi < 0$ there was no pole along the path of integration. In the PV procedure on the other hand, we must have $0 \leq \xi$.

APPENDIX VII

EVALUATION OF $\int_{-\infty}^{\infty} dx/x^2-\xi^2)$

In this Appendix we discuss the evaluation of the integral in the title. The reason for doing this is that in the process we are able to emphasize certain points connected with the δ-function interpretation. In addition, it appears that in some instances the results given in the literature were erroneous. Thus in Ref. 17, this integral is taken to be zero, without specifying in what sense or for what values of ξ is this statement valid.

The integral

$$I(\xi) = \int_{-\infty}^{\infty} \frac{dx}{x^2-\xi^2} \tag{VII-1}$$

has the peculiarity that for $\xi = 0$, its PV is zero as can be immediately verified from the PFD

$$\frac{1}{x^2-\xi^2} = \frac{1}{2\xi}\left[\frac{1}{x-\xi} - \frac{1}{x+\xi}\right]; \quad \xi \neq 0 \tag{VII-2}$$

If ξ can take the value zero, then we must write instead of (VII-2) the more general expression

$$\left.\frac{1}{x^2-\xi^2}\right|_g = \frac{1}{2}\left[\left.\frac{1}{\xi}\right|_p + K\,\delta(\xi)\right]\left[\left.\frac{1}{x-\xi}\right|_p + A\,\delta(x-\xi) - \left.\frac{1}{x+\xi}\right|_p - B\,\delta(x+\xi)\right] \tag{VII-3}$$

where A, B, K are arbitrary constants. Eq. (VII-3) follows from the results of Section II (c).

When we perform the multiplications indicated in (VII-3) we find

$$\left.\frac{1}{x^2-\xi^2}\right|_g = \left.\frac{1}{x^2-\xi^2}\right|_p + K\xi\delta(\xi)\left.\frac{1}{x^2-\xi^2}\right|_p + \frac{K(A-B)}{2}\delta(\xi)\delta(x) + \left.\frac{1}{2\xi}\right|_p (A\delta(x-\xi) - B\delta(x+\xi)) \tag{VII-4}$$

Substituting (VII-4) into (VII-1) we obtain:

$$I^g(\xi) = I^b(\xi) + C_1 \left. \frac{1}{\xi} \right|_P + C_2\delta(\xi) \tag{VII-5}$$

where $C_1 = (A-B)/2$, $C_2 = KC_1$ are again arbitrary constants.

If in (VII-5) we choose $I^p(\xi) = 0$, we have finally for the general determination of $I(\xi)$

$$I^g(\xi) = C_1 \left. \frac{1}{\xi} \right|_P + C_2\delta(\xi) \tag{VII-6}$$

The variable ξ in $1/\xi|_P$ takes all possible values except $\xi = 0$.

A particular value of $I(\xi)$ is given in Ref. 18, namely

$$P\int_{-\infty}^{\infty} \frac{dx}{x^2-\xi^2} = \pi^2\delta(\xi) \tag{VII-7}$$

This result, which corresponds to the choice $C_1=0$ and $C_2=\pi^2$ in (VII-6), is not quite correct in the way it is written in (VII-7). We show below in what sense we can attach this particular determination to $I(\xi)$. Let us take the Fourier transform of $I(\xi)$:

$$\int_{-\infty}^{\infty} e^{-i\lambda\xi} I(\xi)d\xi = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dx \frac{e^{-i\lambda\xi}}{x^2-\xi^2}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\xi \frac{e^{-i\lambda\xi}}{x^2-\xi^2} \tag{VII-8}$$

In (VII-8) we have assumed that inverting the order of integrations is legitimate. The inner integral, over ξ, can be evaluated in the sense of a PV with the aid of eq. (II-22), which yields here

$$P\int_{-\infty}^{\infty} \frac{e^{-i\lambda\xi}}{x^2-\xi^2} d\xi = \frac{\pi\sin(|\lambda|x)}{x} \tag{VII-9}$$

Substituting (VII-9) into (VII-8) we obtain

$$\int_{-\infty}^{\infty} e^{-i\lambda\xi} I(\xi)d\xi = \pi \int_{-\infty}^{\infty} \frac{\sin(|\lambda|x)}{x} dx = \pi^2 \operatorname{sgn}(|\lambda|) \tag{VII-10}$$

Taking now the inverse Fourier transform of (VII-10) we obtain

$$I^P(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{i\lambda\xi}\, \pi^2 \mathrm{sgn}(|\lambda|) = \pi^2 \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda\xi}\, d\lambda\right] = \pi^2\, \delta(\xi) \qquad \text{(VII-11)}$$

All these manipulations show that the P symbol attached to $I(\xi)$ in (VII-7) applies actually to a different integral. Moreover, we have tacitly assumed in (VII-9) that $\lambda \neq 0$. The opposite case, of $\lambda = 0$, brings us back to the original integral I(x). Thus we see that this method of finding a particular determination is fraught with difficulties and uncertainties.

The conclusion we draw from the preceding analysis is that for any particular problem, we should choose for $I(\xi)$ (from eq. (VII-6)) the value yielding an answer, which is correct according to criteria imposed by the original problem.

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